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Choosing within and between Groups

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Abstract

We introduce a theoretical framework in which to study interdependent preferences, where the outcome of others affects the preferences of the decision maker. The dependence may take place in two conceptually different ways, depending on how the decision maker evaluates what the others have. In the first he values his outcome and that of others on the basis of his own utility. In the second, he ranks outcomes according to a social value function. These two different views of the interdependence have separate axiomatic foundations. We then characterize preferences according to the relative importance assigned to social gains and losses, or in other words to pride and envy. Finally, we study a two period economy in which agents have our social preferences. We show how envy leads to conformism in consumption behavior and pride to diversity.

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Contents

"In Silicon Valley, millionaires who don't feel rich," *The New York Times*[1]

1 Introduction

Man is a social animal, as Seneca famously wrote. Who we are, our persona, is shaped by both the private and social consequences of our choices. In contrast, Decision Theory has been mainly concerned with the private side of economic choices: standard preference functionals give no importance to the relative standing of the outcomes of the decision maker relatively to those of his peers. This is in stark contrast with the large empirical literature that emphasizes the importance of relative outcomes in economic choice. In Section 1.4 we will see, for example, how the empirical significance of relative income and consumption has been widely studied, from Dusenberry's early contribution to the many recent works on external habits, the so-called *keeping up the with the Joneses* phenomenon.

The first purpose of this paper is to fill this important gap between theory and empirical evidence by providing a general choice model that takes into account the concern for relative outcomes. We generalize the classic subjective expected utility model by allowing decision makers' preferences to depend on their peers' outcomes. The axiomatic system and the representation are simple, and reduce naturally to the standard theory when the decision maker is indifferent to the outcome of others. We describe the representation in Section 1.1.

How the relative standing of peers' outcomes affect preferences depends on the decision makers' attitudes toward social gains and losses, that is, on their feelings of envy and pride. We call envy (*invidia*) the negative emotion that agents experience when their outcomes fall below those of their peers, and we call pride (*superbia*) the positive emotion that agents experience when they have better outcomes than their peers.[2] Attitudes toward social gains and losses describe the way concern for relative outcomes affect individual preferences. Also these attitudes differ across individuals. Our second purpose is thus to provide the conceptual tools to make meaningful intra personal comparisons ("a person is more proud than envious") and inter personal comparisons ("a decision maker is more envious than another one"). The psychological motivations for the concern for relative outcomes and their main characteristics are discussed in Section 1.2

Our third and final purpose is to provide a link between the features of the preferences that we have identified and the main properties of economic equilibria. For example, in a general two-period economy we provide a link between equilibrium income distribution and the relative weights of envy and pride in agents' preferences. This link is outlined in Section 1.3

1.1 The Representation and its Interpretation

We consider preferences of an agent o. Let $\left(f_o, (f_i)_{i \in I}\right)$ represent the situation in which agent o takes act f_o, while each member i of the agent's reference group takes act f_i. Our decision

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[1]Article by Gary Rivlin, August 5, 2007.

[2]Smith and Kim (2007) review different meanings of envy. They define envy as "an unpleasant and often painful blend of feelings characterized by inferiority, hostility, and resentment caused by a comparison with a person or group of persons who possess something we desire." This is essentially the same definition that we use. Envy and pride are fundamental and specular social emotions, often explicitly considered by religious and social norms, from sumptuary laws (Vincent 1934) to the sixth and seventh deadly sins of the early Christian tradition (Aquaro 2004).

maker then evaluates this situation according to:

$$V\left(f_o, (f_i)_{i \in I}\right) = \int_S u\left(f_o(s)\right) dP(s) + \int_S \varrho\left(v\left(f_o(s)\right), \sum_{i \in I} \delta_{v(f_i(s))}\right) dP(s) \qquad (1)$$

The first term of this representation is familiar. The index $u\left(f_o(s)\right)$ represents the agent's intrinsic utility of the realized outcome $f_o(s)$, P represents his subjective probability over the state space S, and so the first term represents his subjective expected utility from act f_o. The effect on o's welfare of the outcome of the other individuals is reported in the second term. The index $v\left(f_o(s)\right)$ represents the social value o attaches to outcome $f_o(s)$. This index v may or may not be equal to u, according to two completely different interpretations for the concern for relative outcomes that we describe in section 1.2.

Given a profile of acts, agent's peers will get outcomes $(f_i(s))_{i \in I}$ once state s obtains. If o does not care about the identity of who gets the value $v(f_i(s))$, then he will only be interested in the distribution of these values. This distribution is represented by the term $\sum_{i \in I} \delta_{v(f_i(s))}$ in (1) above, where δ_x is the measure giving mass one to x. Finally, the function ϱ is increasing in the first component and stochastically decreasing in the second. This term represents o's satisfaction that derives from the comparison of his outcome with the distribution of outcomes in his reference group. The feeling is experienced *ex-post*, after the realization of the state.

The choice criterion (1) is an ex ante evaluation, combining standard subjective expected utility and the expected ex post envy/pride feeling that decision makers anticipate. That is, in choosing among acts decision makers consider both the private benefit of their choices, $\int_S u\left(f_o(s)\right) dP(s)$, and their social externalities described by $\int_S \varrho\left(v\left(f_o(s)\right), \sum_{i \in I} \delta_{v(f_i(s))}\right) dP(s)$. Standard theory is the special case when the function ϱ is zero, namely when decision makers do not care about possible social externalities of their choices and only the intrinsic properties of the choices' material outcomes matter.

We consider this ex ante compromise as the fundamental trade-off that social decision makers face. This compromise takes a simple, additive, form in (1). This is a very parsimonious extension of standard theory able to deal with concerns for relative outcomes. Behavioral foundation and parsimony are thus two major features of our criterion (1). In contrast, as we detail in Section 10.2, the ad hoc specifications used in empirical work often overlook this key trade-off and focus only relative outcome effects, that is, on the ϱ component of (1).

Finally, observe that for fixed $(f_i)_{i \in I}$ the preference functional (1) represents agent's within group preferences over outcomes, which are conditional on a group having $(f_i)_{i \in I}$. For fixed f_o, the preference functional (1) instead represents between groups preferences, which are conditional on the agent's outcome. Depending on which argument in V is fixed, either f_o or $(f_i)_{i \in I}$, the functional V thus represents preferences between or within groups.

1.2 Envy and Pride

The index ϱ in the representation (1) describes the effect on the decision maker's well being of the social profile of outcomes. The social value of these outcomes is recorded by the index v. If v is equal to the index u, the representation is derived in Theorem 1; if it is different, the representation is derived in Theorem 2. These two different representations correspond to two different views and explanations of the effect of the fortune of others on our preferences. To focus our analysis, we concentrate on envy. We propose two explanations of this key social emotion, based on learning, dominance, and competition.

A Private Emotion An introspective view suggests that when we are envious we consider the outcomes of others, like goods and wealth, thinking how we would enjoy them, evaluating those goods from the point of view of our own utility, and comparing it to the utility that we derive from our own goods and wealth. This interpretation requires $u = v$ in the representation (1). This view of envy points to a possible functional explanation: the painful awareness that others are achieving something we consider enjoyable reminds us that perhaps we are not doing the best possible use of our abilities. Envy is a powerful tool of learning how to deal with uncertainty, by forcing us to evaluate what we have compared to what we could have.

Envy is, from this point of view, the social correspondent of regret. These two emotions are both based on a counterfactual thought. Regret reminds us that we could have done better, had we chosen a course of action that was available to us, but we did not take. Envy reminds us that we could have done better, had we chosen a course of action that was available to us, but someone else actually chose, unlike us. In both cases, we are evaluating the outcome of choices that we did not make from the standpoint of our own utility function; that is, $u = v$ in (1). We regret we did not buy a house that was cheap at some time in which the opportunity presented itself because we like the house: and we envy the house of the neighbor because we like the house. In both cases, we learn that next time we should be more careful and determined in the use of our talent.

A related view of envy motivates the classic theory of social comparison developed by Festinger (1954a, 1954b): People have a drive for a precise evaluation of their own opinion and abilities. An important source of information for such an evaluation is provided by the outcome of others: for example, if I want to know whether obtaining a law degree is hard or easy it is useful to know whether others have succeeded or failed. A corollary of this premise is the similarity hypothesis: individuals will typically be more interested in the outcome of others who are similar, the peers, rather than dissimilar, because the information that we derive on their probability of success is going to be more relevant for the evaluation of our own probability. The theory has been tested and further developed in the last fifty years: the fundamental intuition is that we consider the outcome of others as informative on the nature of the task, and on the relationship between effort and probability of success, just as we consider our previous personal experiences.[3]

Envy is, however, an essentially social emotion. We do care whether the successful outcome is simply a counterfactual thought (as in regret) or the concrete good fortune of someone else. We may feel envy even if we do not like at all the good that the other person has. There must be another reason for envy, a purely social one.

A Social Emotion The search for dominance through competition is a most important force among animals because of the privileged access to resources, most notably food and mates, that status secures to dominant individuals. The organization of societies according to a competition and dominance ranking is thus ubiquitous, extending from plants to ants to primates. For example, plants regulate competition toward kin (Falik, Reides, Gersani, and Novoplansky 2003 and Dudley and File 2007), and examples of hierarchical structures have been documented in insects (Wilson 1971), birds (Schjelderup-Ebbe 1935 and Chase 1982), fishes (Nelissen 1985 and Chase, Bartolomeo and Dugatkin 1994, Chase, Tovey, Spangler-Martin and Manfredonia 2002), and

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[3]See, e.g., Suls, Martin, and Wheeler (2002) for an introduction to recent advances in Social Comparison Theory.

mammals (Greenberg-Cohen, Alkon and Yom-Tov 1994), particularly primates (Maslow 1936, Dunbar 1988, Cheney and Seyfarth 1990, Pereira 1995, Pereira and Fairbanks 2002, Sapolsky 2005).

Quite naturally, competition and dominance feelings play a fundamental role in human societies too, whose members have a very strong preference for higher positions in the social ranking: the proposition has been developed in social psychology, from Maslow (1937) to Hawley (1999) and Sidanius and Pratto (1999). Envy induced by the success of others is the painful awareness that we have lost relative positions in the social ranking. In this view the good that the other is enjoying is not important for the utility it provides and we do not enjoy, but for its cultural/symbolic meaning, that is, for the signal it sends. This signal is important because others, in addition to us, can see it and accordingly change their view on what our current ranking is.

Since it is perceived in a social environment, the way in which it is evaluated has to be social and different from the way in which we privately evaluate it. We may secretly dislike, or fail to appreciate, an abstract painting. But we may proudly display it in our living room if we think that the signal it sends about us (our taste, our wealth, our social network) is valuable. And we may envy someone who has it, even if we would never hang it in our bedroom if we had it. When the effect of the outcome of others is interpreted in this way, the index is a function v, possibly different from the private evaluation function u.

This social index v is as subjective as u: even if they evaluate the outcome of others according to v, individuals have personal views on what society considers important. For example, a specific individual may have a completely wrong view of what peers deem socially important. The perception of what peers consider important, as opposed to what the decision maker values and likes, is taken into account when evaluating peers' outcomes. This is what *subjectively* (as everything else in decision theory) the individual regards as considered *socially* valuable.

1.3   Social Economics

To illustrate the economic scope of our derivation, in Section 9 we study a general two period economy where agents have our social preferences. Our main finding is that in these social economies envy leads to conformism, pride to diversity.

Specifically, we consider an economy with a continuum of identical agents who live for two periods. In the first period agents have an endowment y and choose a consumption c, from which they derive a utility $u(c)$. In the second period they receive a stochastic endowment $Y(s)$, and their consumption is then given by $d(s) = Y(s) + R(y-c)$, from which they derive a discounted expected utility $\beta \mathbb{E}[u(Y + R(y-c))]$.

In every period agents' utility functions have a "social" additional term of the simple form:

$$\gamma(c - \overline{c})$$

where $\overline{c}$ is the average consumption in the economy and γ is an increasing function. In the second period this externality is discounted by the same discount factor β.

When deciding how much to consume in the first period, the agent faces a trade-off: if he increases his consumption today he will increase his relative ranking today, but he will also decrease his standing in the next period. He is thus comparing a positive effect today with a negative effect in the next period. This trade-off (for example, noted in Binder and Pesaran 2001 and Arrow and Dasgupta 2007) points to a crucial feature of the preferences: the relative strength of the effect on individual welfare of being in a dominant or dominated position in the social hierarchy.

To better see how these relative strength affect choices, we consider the two polar cases of pure envy and pure pride. The equilibrium set will be completely different in the two cases: it will be conformist in the case of pure envy (all agents consume the same) and diversified in the pure pride case (identical agents choose a different consumption).

Agents with pure envy preferences only care about the situation in which their consumption is below the average value. For example take

$$\gamma(x) = \theta x^-,$$

where $x^- \equiv \min(x, 0)$. Since the function is concave, the overall program of each agent is concave, and so the equilibrium is symmetric: all agents choose the same consumption.

In contrast, agents with pure pride preferences have, for example,

$$\gamma(x) = \theta x^+,$$

where $x^+ \equiv \max(x, 0)$. This function is convex, something that completely changes the structure of the equilibrium set. The equilibrium can only be non-symmetric: although agents are identical, they will choose different consumptions. Some will choose to have a dominant position in the current period, at the expense of a dominated one in the future, and others will choose the opposite.

Summing up, envy is, loosely speaking, a concave externality while pride is a convex one. The resulting equilibria are then qualitatively very different in the two cases. This is a novel insight of our analysis that was made possible by the axiomatic analysis of the earlier part of the paper.

We close by observing how the social nature of consumption emerges in these social economies. In fact, even though we consider economies with a continuum of individually negligible agents, whose behavior does not affect per se the outcomes of the other agents, still Nash equilibrium is the natural solution concept to use. Here, agents' interact through the consumption externalities caused by the social dimension of their decisions; the social economy is in equilibrium when all these consumption externalities balance each others.

1.4   Related Literature

The modern economic formulation of the idea that the welfare of an agent depends on the relative as well as the absolute consumption is usually attributed to Veblen (1899):

> ... soon as the possession of property becomes the basis of popular esteem, therefore, it becomes also a requisite to the complacency which we call self-respect. In any community where goods are held in severalty it is necessary, in order to his own peace of mind, that an individual should possess as large a portion of goods as others with whom he is accustomed to class himself; and it is extremely gratifying to possess something more than others ...

Fifty years after *The Theory of the Leisure Class*, social psychology dealt with the issue of social comparison with the works of Festinger (1954a, 1954b). As we mentioned, the focus of the theory is orthogonal to that of Veblen: the comparison with others is motivated by learning, and the outcome of the others is relevant to us only because it provides information that may be useful in improving our performance.

Veblen and Festinger provided the first instances of the two views of social emotions. Our work is an attempt to provide a structure in which these two views can be compared and experimentally tested. Veblen's view has been dominant or even exclusive in inspiring research in economics. We hope that our paper may help in restoring a more balanced view.

As we just observed, the Veblenian intuition that an agent's well-being is determined not only by the intrinsic utility of his material consumption, but also by his relative standing in the society or in his peer group had a huge impact on the socioeconomic thought (e.g., Duesenberry 1949, Easterlin 1974, Frank 1985, and Schor 1998) and the phenomenon called *keeping up with the Joneses* has been heating economic debate for the last two decades.

The significance of one's relative outcome standing has been widely studied in the economics and psychology of subjective well-being (e.g., Easterlin 1995 and Frey and Stutzer 2002, and the references therein) and there is a large body of direct and indirect empirical evidence in support of this fundamental hypothesis (e.g., Easterlin 1974, van de Stadt, Kapteyn, and van de Geer 1985, Tomes 1986, Clark and Oswald 1996, McBride 2001, Zizzo and Oswald 2001, Luttmer 2005, Ravina 2005, and Dynan and Ravina 2007).

At the same time, the introduction of agents' concerns for relative outcomes, especially in consumption and income, into economic models has been shown to carry serious implications cutting across different fields such as demand analysis (e.g., Leibenstein 1950, Gaertner 1974, Pollak 1976, Kapteyn, Van de Geer, Van de Stadt, and Wansbeek 1997, and Binder and Pesaran 2001), taxation and expenditure policy (e.g., Boskin and Sheshinski 1978, Layard 1980, Oswald 1983, Ng 1987, Villar 1988, Blomquist 1993, Ljungqvist and Uhlig 2000, and Abel 2005), equilibrium and asset pricing (e.g., Abel 1990, Galí 1994, Abel 1999, Campbell and Cochrane 1999, Chan and Kogan 2002, and Dupor and Liu 2003), labor search and wage determination (e.g., Frank 1984a and 1984b, Akerlof and Yellen 1990, Neumark and Postlewaite 1998, and Bowles and Park 2005), growth (e.g., Carroll, Overland, and Weil 1997, Corneo and Jeanne 2001, and Liu and Turnovsky 2005), and corporate investments (e.g., Goel and Thakor 2005).[4]

Theoretical Work   Despite their intuitive appeal and empirical relevance, there is surprisingly very little theoretical work on other-regarding preferences, and, in particular, on preferences of agents who care about relative outcomes. We are only aware of the works of Ok and Koçkesen (2000), Gilboa and Schmeidler (2001), Gul and Pesendorfer (2005), Karni and Safra (2002), Neilson and Stowe (2004), Neilson (2006), Sandbu (2005), and Segal and Sobel (2007). Among them, Ok and Koçkesen (2000) is the article closest to ours. They consider negative interdependent preferences over income distributions x and provide an elegant axiomatization of the relative income criterion $x_o f(x_o/\overline{x})$, where $\overline{x}$ is the society average income and f is a strictly increasing function. In deriving their criterion, Ok and Koçkesen (2000) emphasize the distinction in agents' preferences over income distributions between relative and individual income effects, modelled by $f(x_o/\overline{x})$ and x_o, respectively. This distinction is a special instance of the general trade-off between private benefits and social externalities we discussed before. In particular, the ordinal logarithmic transformation of the criterion $x_o f(x_o/\overline{x})$ is a special case of Theorem

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[4]Other rank/status concerns have been considered by many authors. See, for example, the following articles (and the references therein): Weiss (1976), Jones (1984), Basu (1989), Baumol (1990), Robson (1992), Cole, Mailath, and Postlewaite (1992), Fershtman and Weiss (1993), Bernheim (1994), Ireland (1994), Kolm (1995), Mui (1995), Pesendorfer (1995), Bagwell and Bernheim (1996), Fershtman, Murphy, and Weiss (1996), Akerlof (1997), Weiss and Fershtman (1998), Ball, Eckel, Grossman, and Zame (2001), Brock and Durlauf (2001), Sameulson (2004), Becker, Murphy, and Werning (2005), Rayo and Becker (2007).

3 below, in which we axiomatize the following version of the general criterion (1)

$$V\left(f_o,(f_i)_{i\in I}\right) = \int_S u\left(f_o\left(s\right)\right) dP\left(s\right) + \int_S \varrho\left(v\left(f_o\left(s\right)\right), \frac{1}{|I|}\sum_{i\in I} v\left(f_i\left(s\right)\right)\right) dP\left(s\right),$$

where decision makers only care about average outcomes.

A different theoretical approach has been proposed by Michael and Becker (1973), Becker (1974), and Stigler and Becker (1977) (see Lancaster 1966 for a related approach), in which utility analysis is reformulated by considering basic needs as arguments of agents' objective functions, in place of market consumption goods.[5] The latter are viewed as inputs in household production functions, whose outputs are the basic needs. In contrast, in our economic applications of Section 9 we still regard market consumption goods as arguments of our objective functions, and the role of emotions is to shape the objective functions' form.

Finally, the theoretical studies of Samuelson (2004) and Rayo and Becker (2007b) investigate the emergence of relative outcome concerns from an evolutionary point of view. They show how it can be evolutionary optimal to build relative outcome effects directly into the utility functions. In particular, Samuelson proposes an information based explanation, where peers' outcomes are regarded as an information source on the environment's uncertainties, while in Rayo and Becker's analysis relative concerns emerge from the maximization of relative rather than absolute measures of fitness.

Other Social Emotions   Other strands of literature consider different social emotions, like, for example, altruism and related feelings (e.g., desire for reciprocity and fairness) that arise from concerns about peers' well being (see, e.g., Levine, 1998, Fehr and Schmidt, 1999, Bolton and Ockenfels, 2000, Charness and Rabin, 2002, as well as the surveys of Fehr and Fischbacher, 2002, and Sobel, 2005). Although economic social emotions are related as they are all ultimately determined by the peers' material outcomes, altruism and other emotions caused by the peers' well being are conceptually different from the emotions that we study in this paper. In particular, the behavioral axioms that we introduce to derive the preference functional (1) are motivated and interpreted by relative outcome concerns, and not by other concerns determined by peers' outcomes. No element in (1) represents, at least explicitly, the decision makers perceptions of their peers well being.

That said, our analysis provides a framework in which it is possible to model, through suitable modifications/adaptations of our behavioral axioms, other economically relevant social emotions. For example, to further illustrate the scope of our framework in Section 10.6 we briefly show how to modify our behavioral axioms in order to model inequity aversion à la Fehr and Schmidt (1999).

1.5   Organization

The rest of the paper is organized as follows. Section 2 presents some preliminary notions, used in Section 3 to state our basic axioms. Section 4 and 5 contain our main results; in particular, in Section 4 we prove the private utility representation, while in Section 5 we derive the social one. Section 6 considers few special cases of our representations. Sections 7 and 8 provide behaviorally based conditions on the shapes of the elements of the representations. Section 9

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[5]For example, Becker (1974) considers an objective function with the "need of distinction" as an argument (with its associated emotion of envy). See Sobel (2005) for an overview and discussion of Becker's approach.

illustrates our representation with an economic applications. Finally, Section 10 discusses at length some further conceptual issues relevant for our analysis that we did not discuss in the Introduction, including a regret interpretation of our setting. All proofs are collected in the Appendix.

2 Preliminaries

2.1 Setup

We consider a standard Anscombe and Aumann (1963) style setting. Its basic elements are a set S of *states of nature*, an algebra Σ of subsets of S called *events*, and a convex set C of *consequences*.

We denote by o a given *agent*, our protagonist, and by N the non-empty, possibly infinite, set of all agents in o's world that are different from o himself, that is, the set of all his possible peers (the "Joneses," as they are often called in the literature).

We denote by $\wp(N)$ the set of all finite subsets of N; notice that $\emptyset \in \wp(N)$. Throughout the paper, I denotes an element of $\wp(N)$, even where not stated explicitly (except in Section 9). For every I, we denote by I_o the set $I \cup \{o\}$; similarly, if j does not belong to I, we denote by I_j the set $I \cup \{j\}$.

A *(simple) act* is a Σ-measurable and finite-valued function from S to C. We denote by $\mathcal{A}$ the set of all acts and by $\mathcal{A}_i$ the set of all acts available to agent $i \in N_o$; finally

$$\mathcal{F} = \left\{ \left(f_o, (f_i)_{i \in I}\right) : I \in \wp(N), \ f_o \in \mathcal{A}_o, \text{ and } f_i \in \mathcal{A}_i \text{ for each } i \in I \right\}$$

is the set of all *act profiles*. Each act profile $f = \left(f_o, (f_i)_{i \in I}\right)$ describes the situation in which o selects act f_o and his peers in I select the acts f_i.

When I is the empty set (i.e., o has no reference group of peers), we have $f = (f_o)$ and we often will just write f_o to denote such a "Robinson Crusoe" profile.

Here it is important to observe how the outcomes obtained by the agents in each state of nature do not depend on the acts chosen by the other agents.

The constant act taking value c in all states is still denoted by c. With the usual slight abuse of notation, we thus identify C with the subset of the constant acts. The set of acts profiles consisting of constant acts is denoted by $\mathcal{X}$, that is,

$$\mathcal{X} = \left\{ \left(x_o, (x_i)_{i \in I}\right) : I \in \wp(N), \ x_o \in C \cap \mathcal{A}_o, \text{ and } x_i \in C \cap \mathcal{A}_i \text{ for each } i \in I \right\}.$$

Clearly, $\mathcal{X} \subseteq \mathcal{F}$ and we denote by c_{I_o} an element $x = \left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$ such that $x_i = c$ for all $i \in I_o$.[6]

Throughout the paper (except in Section 9) we make the following structural assumption.

Assumption. $\mathcal{A}_o = \mathcal{A}$ and each $\mathcal{A}_i$ contains all constant acts.

In other words, we assume that o can select any act and that his peers can, at least, select any constant act. This latter condition on peers implies that the consequences profile at s, $f(s) = \left(f_o(s), (f_i(s))_{i \in I}\right)$, belong to $\mathcal{X}$ for all $f = \left(f_o, (f_i)_{i \in I}\right) \in \mathcal{F}$ and all $s \in S$.

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[6]Similarly, c_I denotes a constant $(x_i)_{i \in I}$.

2.2 Distributions

We now introduce distributions, which play a key role in the paper. Let A be any set, for example a set of outcomes or payoffs. If $I \in \wp(N)$ is not empty, set $A^I = \times_{i \in I} A$. Given a vector $\mathbf{e} = (e_i)_{i \in I} \in A^I$, we denote by $\mu_{\mathbf{e}} = \sum_{i \in I} \delta_{e_i}$ the distribution of $\mathbf{e}$. That is, for all $a \in A$,

$$\mu_{\mathbf{e}}(a) = \sum_{i \in I} \delta_{e_i}(a) = |\{i \in I : e_i = a\}|.$$

In other words, $\mu_{\mathbf{e}}(a)$ is the number of indices i, that is, of agents, that get the same element a of A under the allocation $\mathbf{e}$.

Let $\mathcal{M}(A)$ be the collection of all positive integer measures μ on A with finite support and such that $\mu(A) \leq |N|$. That is,[7]

$$\mathcal{M}(A) = \left\{ \sum_{i \in I} \delta_{e_i} : I \in \wp(N) \text{ and } e_i \in A \text{ for all } i \in I \right\}.$$

In other words, $\mathcal{M}(A)$ is the set of all possible distributions of vectors $\mathbf{e} = (e_i)_{i \in I}$ in A^I, while I ranges in $\wp(N)$.

Set $\mathrm{pim}(A) = A \times \mathcal{M}(A)$. For example, if A is a set of payoffs, pairs $(z, \mu) \in \mathrm{pim}(A)$ are understood to be of the form

(payoff of o, distribution of peers' payoffs).

A function $\varrho : \mathrm{pim}(A) \to \mathbb{R}$ is *diago-null* if

$$\varrho(z, n\delta_z) = 0, \qquad \forall z \in A, 0 \leq n \leq |N|. \tag{2}$$

For example, when A is a set of outcomes, a diago-null function ϱ is zero whenever o and all his peers are getting the same outcome.

When $A \subseteq \mathbb{R}$, the order structure of $\mathbb{R}$ makes it possible to introduce monotone distribution functions. Specifically, given $\mathbf{e} \in \mathbb{R}^I$,

$$F_{\mathbf{e}}(t) = \mu_{\mathbf{e}}(-\infty, t] = |\{i \in I : e_i \leq t\}|, \text{ and}$$
$$G_{\mathbf{e}}(t) = \mu_{\mathbf{e}}(t, \infty) = |\{i \in I : e_i > t\}| = |I| - F_{\mathbf{e}}(t)$$

are the increasing and decreasing distribution functions of $\mathbf{e}$, respectively.[8]

Given two vectors $\mathbf{a} = (a_i)_{i \in I} \in \mathbb{R}^I$ and $\mathbf{b} = (b_j)_{j \in J} \in \mathbb{R}^J$, we say that:

(i) $\mu_{\mathbf{a}}$ *upper dominates* $\mu_{\mathbf{b}}$ if $G_{\mathbf{a}}(t) \geq G_{\mathbf{b}}(t)$ for all $t \in \mathbb{R}$,

(ii) $\mu_{\mathbf{a}}$ *lower dominates* $\mu_{\mathbf{b}}$ if $F_{\mathbf{a}}(t) \leq F_{\mathbf{b}}(t)$ for all $t \in \mathbb{R}$,

(iii) $\mu_{\mathbf{a}}$ *stochastically dominates* $\mu_{\mathbf{b}}$ if $\mu_{\mathbf{a}}$ both upper and lower dominates $\mu_{\mathbf{b}}$.

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[7]We adopt the convention that any sum of no summands (i.e., over the empty set) is zero.

[8]When $I = \emptyset$, $\mu_{\mathbf{e}} = 0$ and so $F_{\mathbf{e}} = G_{\mathbf{e}} = 0$.

3 The Basic Axioms

Our main primitive notion is a binary relation $\succsim_o$ on the set $\mathcal{F}$ that describes o's preferences. As anticipated in the Introduction, the ranking

$$\left(f_o, (f_i)_{i \in I}\right) \succsim_o \left(g_o, (g_j)_{j \in J}\right)$$

is the agent's ranking among societies (peer groups). To ease notation, we will often just write $\succsim$ instead of $\succsim_o$.

Axiom A. 1 (Weak Order) $\succsim$ *is nontrivial, complete, and transitive.*

Axiom A. 2 (Monotonicity) *Let $f, g \in \mathcal{F}$. If $f(s) \succsim g(s)$ for all s in S, then $f \succsim g$.*

Axiom A. 3 (Archimedean) *For all $\left(f_o, (f_i)_{i \in I}\right)$ in $\mathcal{F}$, there exist $\underline{c}$ and $\bar{c}$ in C such that*

$$\underline{c}_{I_o} \precsim \left(f_o, (f_i)_{i \in I}\right) \quad and \quad \left(f_o, (f_i)_{i \in I}\right) \precsim \bar{c}_{I_o}.$$

Moreover, if the above relations are both strict, there exist $\alpha, \beta \in (0, 1)$ such that

$$(\alpha \underline{c} + (1 - \alpha)\bar{c})_{I_o} \prec \left(f_o, (f_i)_{i \in I}\right) \quad and \quad \left(f_o, (f_i)_{i \in I}\right) \prec (\beta \underline{c} + (1 - \beta)\bar{c})_{I_o}.$$

These first three axioms are standard. The Monotonicity Axiom requires that if an act profile f is, state by state, better than another act profile g, then $f \succsim g$. Note that in each state the comparison is between social allocations, that is, between elements of $\mathcal{X}$. In each state, o is thus comparing outcome profiles, not just his own outcomes. In Section 10.3 we will further discuss Axiom 2.

Axiom A. 4 (Independence) *Let α in $(0, 1)$ and f_o, g_o, h_o in $\mathcal{A}_o$. If $(f_o) \succ (g_o)$, then*

$$(\alpha f_o + (1 - \alpha) h_o) \succ (\alpha g_o + (1 - \alpha) h_o).$$

This is a classic *independence axiom*, which we only require on "solo" preferences, with no peers.

Axiom A. 5 (Conformistic Indifference) $c_{I_o} \sim c_{I_o \cup \{j\}}$ *for all c in C, I in $\wp(N)$, and j not in I.*

According to this axiom, for agent o it does not matter if to an "egalitarian" group, where everybody has the same outcome c, is added a further peer with, again, outcome c.

Axiom A.5 thus describes a very simple form of the trade-off, from the standpoint of the preferences of o, between an increase in the size of the society and the change in the outcome necessary to keep him indifferent. In the representation this axiom translates into the condition that the externality function ϱ is zero when all members of the group have the same outcome.

Different trade-offs have a similar axiomatization. For example, if o prefers, for the same outcome c, a smaller society, then a similar axiom would require that, for some improvement over c, he would feel indifferent between the smaller society with a less preferred outcome and a larger one with better common outcome. With this more general axiom, the externality function would also depend on the size of the group.[9]

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[9]That is, in (2) we would have $\varrho(z, n\delta_z) = \phi(n)$, where $\phi : \mathbb{N} \to \mathbb{R}$ is a suitable decreasing function.

Axiom A.5 per se is especially appealing for large groups; in any case, we regard it as a transparent and reasonable simplifying assumption, whose weakening would complicate the derivation without a comparable benefit for the interpretation.

The next final basic axiom is an anonymity condition, which assumes that decision makers do not care about the identity of who, among their peers, gets a given outcome. This condition requires that only the distribution of outcomes matters, without any role for possible special ties that decision makers may have with some of their peers. This allows to study relative outcomes effects in "purity," without other concerns intruding into the analysis.

Axiom A. 6 (Anonimity) *Let* $\left(x_o, (x_i)_{i \in I}\right), \left(x_o, (y_j)_{j \in J}\right)$ *in* $\mathcal{X}$. *If there is a bijection* $\pi : J \to I$ *such that* $y_j = x_{\pi(j)}$ *for all* $j \in J$, *then* $\left(x_o, (x_i)_{i \in I}\right) \sim \left(x_o, (y_j)_{j \in J}\right)$.

The basic axioms A.1-A.6 lead to our basic representation.

Lemma 1 *A binary relation* $\succsim$ *on* $\mathcal{F}$ *satisfies Axioms A.1-A.6 if and only if there exist a non-constant affine function* $u : C \to \mathbb{R}$, *a diago-null function* $\varrho : \mathrm{pim}\,(C) \to \mathbb{R}$, *and a probability* P *on* Σ *such that*

$$V (f) = \int_S u \left(f_o (s)\right) dP (s) + \int_S \varrho \left(f_o (s), \sum_{i \in I} \delta_{f_i(s)}\right) dP (s) \qquad (3)$$

represents $\succsim$ *and satisfies* $V (\mathcal{F}) = u (C)$.

Moreover, $\left(\hat{u}, \hat{\varrho}, \hat{P}\right)$ *is another representation of* $\succsim$ *in the above sense if and only if* $\hat{P} = P$ *and there exist* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha > 0$ *such that* $\hat{u} = \alpha u + \beta$ *and* $\hat{\varrho} = \alpha \varrho$.

In this primitive, "mother," representation relative outcome concerns are captured by the externality function $\varrho : \mathrm{pim}\,(C) \to \mathbb{R}$, which depends on both agent's o own outcome $f_o (s)$ and on the distribution $\sum_{i \in I} \delta_{f_i(s)}$ of peers' outcome. In fact, a pair $(z, \mu) \in \mathrm{pim}\,(C)$ reads as

(outcome of o, distribution of peers' outcomes).

All representations in the paper build on the basic representation (3) and they will be characterized by specific properties of the function ϱ, such as suitable monotonicity properties in its arguments.

Finally, observe that Axioms A.1-A.5 guarantee that, for each $\left(f_o, (f_i)_{i \in I}\right) \in \mathcal{F}$, there exists a $c_o \in C$ such that $\left(f_o, (f_i)_{i \in I}\right) \sim (c_o)$. Such element c_o will be denoted by $c \left(f_o, (f_i)_{i \in I}\right)$.[10]

4   The Private Utility Representation

In this section we present our first representation, which models the private emotion discussed in the Introduction.

The basic Axioms A.1-A.6 are common to our two main representations, the "private" and the, more general, "social." The next two axioms are, instead, peculiar to the private representation. They only involve deterministic act profiles, that is, elements of $\mathcal{X}$.

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[10]The existence of c_o is proved in Lemma 9 of Appendix 11.4, which also shows that Axioms A.1-A.5 give a first simple representation.

Axiom B. 1 (Negative Dependence) *If $\bar{c} \succsim \underline{c}$, then*

$$\left(x_o, (x_i)_{i \in I}, \underline{c}_{\{j\}}\right) \succsim \left(x_o, (x_i)_{i \in I}, \bar{c}_{\{j\}}\right) \tag{4}$$

for all $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$ and $j \notin I$.

Axiom B.1 is a key behavioral condition because it captures the negative dependence of agent o welfare on his peers' outcomes. In fact, according to Axiom B.1 the decision maker o prefers, *ceteris paribus*, that a given peer j gets an outcome that he regards less valuable.

Axiom B. 2 (Comparative Preference) *Let $\left(x_o, (x_i)_{i \in I}\right), \left(y_o, (x_i)_{i \in I}\right) \in \mathcal{X}$. If $x_o \succsim y_o$, then*

$$\frac{1}{2} c\left(x_o, (x_i)_{i \in I}\right) + \frac{1}{2} y_o \succsim \frac{1}{2} x_o + \frac{1}{2} c\left(y_o, (x_i)_{i \in I}\right).$$

Axiom B.2 is based on the idea that the presence of a society stresses the perceived differences in consumption. In fact, interpreting x_o as a gain and y_o as a loss, the idea is that winning in front of a society is better than winning alone, losing alone is better than loosing in front of a society, and, "hence," a fifty-fifty randomization of the better alternatives is preferred to a fifty-fifty randomization of the worse ones.

We can now state the private utility representation, where we use the notation introduced in Section 2.2.

Theorem 1 *A binary relation $\succsim$ on $\mathcal{F}$ satisfies Axioms A.1-A.6 and B.1-B.2 if and only if there exist a non-constant affine function $u : C \to \mathbb{R}$, a diago-null function $\varrho : \mathrm{pim}\,(u\,(C)) \to \mathbb{R}$ increasing in the first component and decreasing (w.r.t. stochastic dominance) in the second one, and a probability P on Σ such that*

$$V\left(f_o, (f_i)_{i \in I}\right) = \int_S u\left(f_o\left(s\right)\right) dP\left(s\right) + \int_S \varrho\left(u\left(f_o\left(s\right)\right), \sum_{i \in I} \delta_{u(f_i(s))}\right) dP\left(s\right) \tag{5}$$

represents $\succsim$ and satisfies $V\left(\mathcal{F}\right) = u\left(C\right)$.

In this representation the externality function ϱ of (3) takes a special form in which outcome profiles $\left(f_o\left(s\right), (f_i\left(s\right))_{i \in I}\right)$ are evaluated via agent's o utility function u. In particular, ϱ depends on both agent o's own payoff $u\left(f_o\left(s\right)\right)$ and on the distribution $\sum_{i \in I} \delta_{u(f_i(s))}$ of peers' outcome, evaluated via u. This dependence is increasing in o's payoff and decreasing (w.r.t. stochastic dominance) in the peers' outcome distribution. This reflects the negative dependence behaviorally modelled by Axiom B.1.

The preferences described by Theorem 1 can be represented by a triplet (u, ϱ, P). Next we give the uniqueness properties of this representation.

Proposition 1 *Two triplets (u, ϱ, P) and $(\hat{u}, \hat{\varrho}, \hat{P})$ represent the same relation $\succsim$ as in Theorem 1 if and only if $\hat{P} = P$ and there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that $\hat{u} = \alpha u + \beta$, and*

$$\hat{\varrho}\left(z, \sum_{i \in I} \delta_{z_i}\right) = \alpha \varrho\left(\alpha^{-1}\left(z - \beta\right), \sum_{i \in I} \delta_{\alpha^{-1}(z_i - \beta)}\right),$$

for all $\left(z, \sum_{i \in I} \delta_{z_i}\right) \in \mathrm{pim}\,(\hat{u}\,(C))$.

5 The Social Value Representation

5.1 An Induced Order

Turn now to the possibility that agents might experience envy/pride feelings because of the outcomes' symbolic value. In fact, an object may be serviceable for the utility it provides to the user abstracting from the social signal it sends; for instance, if the object is used completely in private. But the social value is a different evaluation.

To illustrate, consider the famous "silver spoon" example of Veblen (1899), which clearly brings out the contrast between use and symbolic values of objects:

> A hand-wrought silver spoon, of a commercial value of some ten to twenty dollars, is not ordinarily more serviceable – in the first sense of the word – than a machine-made spoon of the same material. It may not even be more serviceable than a machine-made spoon of some "base" metal, such as aluminum, the value of which may be no more than some ten to twenty cents.

The conceptual structure we have developed so far allows us to make more precise and behaviorally founded the classic Veblenian distinction. Specifically, we formalize this idea by introducing an induced preference $\overset{\cdot}{\succsim}$ on C, which will be represented by a social value function v.

Definition 1 *Given any $\underline{c}, \bar{c} \in C$, say that*

$$\bar{c} \overset{\cdot}{\succsim} \underline{c} \tag{6}$$

if

$$\left(x_o, (x_i)_{i \in I}, \underline{c}_{\{j\}} \right) \succsim \left(x_o, (x_i)_{i \in I}, \bar{c}_{\{j\}} \right) \tag{7}$$

for all $\left(x_o, (x_i)_{i \in I} \right) \in \mathcal{X}$ and $j \notin I$.

In other words, we have $\bar{c} \overset{\cdot}{\succsim} \underline{c}$ when in all possible societies to which the decision maker can belong, he always prefers that, *ceteris paribus*, a given peer has $\underline{c}$ rather than $\bar{c}$.

In particular, only peer j's outcome changes in the comparison (7), while both the decision maker's own outcome x_o and all other peer's outcomes $(x_i)_{i \in I}$ remain the same. The ranking (7) thus reveals through choice behavior a negative outcome externality of j on o.

This negative externality can be due to the private emotion we discussed before; in this case Axiom B.1 holds and the rankings $\succsim$ and $\overset{\cdot}{\succsim}$ are then easily seen to agree on C (i.e., $u = v$ in the representation). More generally, however, this externality can be due to a cultural/symbolic aspect of j's outcome. For instance, the Veblen silver and aluminum spoons are presumably ranked indifferent by $\succsim$, but not by $\overset{\cdot}{\succsim}$. That is, they have similar u values, but different v values.

Summing up, we interpret $\bar{c} \overset{\cdot}{\succsim} \underline{c}$ as revealing, via choice behavior, that our envious/proud decision maker regards outcome $\bar{c}$ to be more socially valuable than $\underline{c}$. If $\succsim$ and $\overset{\cdot}{\succsim}$ do not agree on C, this can be properly attributed to the outcomes' symbolic value.

Remark 1 The relation $\overset{\cdot}{\succsim}$ is trivial for conventional, asocial, decision makers because for them it always holds

$$\left(x_o, (x_i)_{i \in I}, \underline{c}_{\{j\}} \right) \sim \left(x_o, (x_i)_{i \in I}, \bar{c}_{\{j\}} \right) \sim (x_o).$$

That is, asocial decision makers are characterized by the general social indifference $\bar{c} \overset{\cdot}{\sim} \underline{c}$ for all $\bar{c}, \underline{c} \in C$.

5.2 Representation

We now present few behavioral (and so testable) axioms needed for the social representation. We begin with few simple axioms on the primitive preference $\succsim$ that will make the induced preference $\dot{\succsim}$ an independent and nontrivial weak order on C.

Axiom A. 7 (Group Invariance) *Given any $c, d \in C$, if*

$$(x_o, (x_i)_{i \in I}, d_{\{j\}}) \succ (x_o, (x_i)_{i \in I}, c_{\{j\}}) \tag{8}$$

for some $(x_o, (x_i)_{i \in I}) \in \mathcal{X}$ and $j \notin I$, then there is no other $(x_o, (x_i)_{i \in I}) \in \mathcal{X}$ and $j \notin I$ such that $(x_o, (x_i)_{i \in I}, c_{\{j\}}) \succ (x_o, (x_i)_{i \in I}, d_{\{j\}})$.

This axiom requires that the decision maker be consistent across groups in his social ranking of outcomes. This ranking is thus "absolute" and group invariant, that is, it does not depend on the particular peers' group in which the decision maker happens to make the comparison (8). In terms of the representation, Axiom 7 implies that the function v does not depend on I.

Axiom 7 can be regarded as a group anonymity axiom, that is, it does not matter the particular group where a choice is made. Like the anonymity Axiom A.6, this condition guarantees that only outcomes per se matter and it thus allows us to study in purity the relative outcomes effects, our main object of interest.

The following axiom guarantees that the preference $\dot{\succsim}$ is nontrivial.

Axiom A. 8 (Nontriviality) *There are $c, d \in C$, $(x_o, (x_i)_{i \in I}) \in \mathcal{X}$ and $j \notin I$ such that*

$$(x_o, (x_i)_{i \in I}, d_{\{j\}}) \succ (x_o, (x_i)_{i \in I}, c_{\{j\}}). \tag{9}$$

The next two axioms just require standard independence and Archimedean conditions with respect to a given peer j's outcome. To ease notation, $c\alpha d$ denotes $(1 - \alpha) c + \alpha d$.

Axiom A. 9 (Outcome Independence) *For all $c, d, e \in C$, $(x_o, (x_i)_{i \in I}) \in \mathcal{X}$, and $j \notin I$ we have*

$$(x_o, (x_i)_{i \in I}, c_{\{j\}}) \succsim (x_o, (x_i)_{i \in I}, d_{\{j\}}) \implies (x_o, (x_i)_{i \in I}, c\alpha e_{\{j\}}) \succsim (x_o, (x_i)_{i \in I}, d\alpha e_{\{j\}}),$$

for all $\alpha \in (0, 1)$.

Axiom A. 10 (Outcome Archimedean) *If $c, d, e \in C$, $(x_o, (x_i)_{i \in I}) \in \mathcal{X}$, $j \notin I$, and*

$$(x_o, (x_i)_{i \in I}, c_{\{j\}}) \succ (x_o, (x_i)_{i \in I}, d_{\{j\}}) \succ (x_o, (x_i)_{i \in I}, e_{\{j\}}),$$

then there exist $\alpha, \beta \in (0, 1)$ such that

$$(x_o, (x_i)_{i \in I}, c\alpha e_{\{j\}}) \succ (x_o, (x_i)_{i \in I}, d_{\{j\}}) \succ (x_o, (x_i)_{i \in I}, c\beta e_{\{j\}}).$$

Axioms A.7-A. 10 induce the following properties on $\dot{\succsim}$.

Lemma 2 *The preference relation $\succsim$ satisfies axioms A.1 and A.7-A.10 if and only if $\dot{\succsim}$ is complete, transitive, nontrivial, Archimedean, and independent.*

The final axiom we need for the social representation is simply the social version of Axiom B.2.

Axiom A. 11 (Social Comparative Preference) *Let* $\left(x_o, (x_i)_{i\in I}\right), \left(y_o, (x_i)_{i\in I}\right)$ *in* $\mathcal{X}$ *If* $x_o \overset{\cdot}{\succsim} y_o$, *then*

$$\frac{1}{2}c\left(x_o, (x_i)_{i\in I}\right) + \frac{1}{2}y_o \succsim \frac{1}{2}x_o + \frac{1}{2}c\left(y_o, (x_i)_{i\in I}\right)$$

We can now state our more general representation result.

Theorem 2 *A binary relation* $\succsim$ *on* $\mathcal{F}$ *satisfies Axioms A.1-A.11 if and only if there exist two non-constant affine functions* $u, v : C \to \mathbb{R}$, *a diago-null function* $\varrho : \text{pim}(v(C)) \to \mathbb{R}$ *increasing in the first component and decreasing (w.r.t. stochastic dominance) in the second one, and a probability* P *on* Σ, *such that* v *represents* $\overset{\cdot}{\succsim}$ *and*

$$V\left(f_o, (f_i)_{i\in I}\right) = \int_S u\left(f_o(s)\right) dP(s) + \int_S \varrho\left(v\left(f_o(s)\right), \sum_{i\in I}\delta_{v(f_i(s))}\right) dP(s) \qquad (10)$$

represents $\succsim$ *and satisfies* $V(\mathcal{F}) = u(C)$.

Relative to the private representation (5), there is now a non-constant affine function $v : C \to \mathbb{R}$ that represents $\overset{\cdot}{\succsim}$ and so quantifies the social emotion. The function v replaces u in the externality function ϱ, and so here agent o evaluates with v both his own payoff and the peers' outcome. Like u, also v is a purely subjective construct because $\overset{\cdot}{\succsim}$ is derived from the subjective preference $\succsim$. As such, it may depend solely on subjective considerations.

The preferences described by Theorem 2 are thus represented by a quadruple (u, v, ϱ, P). Next we give the uniqueness properties of this representation.

Proposition 2 *Two quadruples* (u, v, ϱ, P) *and* $(\hat{u}, \hat{v}, \hat{\varrho}, \hat{P})$ *represent the same relations* $\succsim$ *and* $\overset{\cdot}{\succsim}$ *as in Theorem 2 if and only if* $\hat{P} = P$ *and there exist* $\alpha, \beta, \dot{\alpha}, \dot{\beta} \in \mathbb{R}$ *with* $\alpha, \dot{\alpha} > 0$ *such that* $\hat{u} = \alpha u + \beta$, $\hat{v} = \dot{\alpha}v + \dot{\beta}$, *and*

$$\hat{\varrho}\left(z, \sum_{i\in I}\delta_{z_i}\right) = \alpha\varrho\left(\dot{\alpha}^{-1}\left(z - \dot{\beta}\right), \sum_{i\in I}\delta_{\dot{\alpha}^{-1}(z_i-\dot{\beta})}\right)$$

for all $\left(z, \sum_{i\in I}\delta_{z_i}\right) \in \text{pim}(\hat{v}(C))$.

5.3 Private vs Social

The fact that the preference functional (5) in Theorem 1 is a special case of (10) in Theorem 2 might suggest that Theorem 1 is a special case of Theorem 2. Because of the requirement in Theorem 2 that v represents $\overset{\cdot}{\succsim}$, this is true provided u also represents $\overset{\cdot}{\succsim}$. That is, provided $\succsim$ and $\overset{\cdot}{\succsim}$ agree on C.

Notice that Axiom B.1 guarantees that $\succsim$ implies $\overset{\cdot}{\succsim}$. The converse implication is obtained by strengthening Axiom B.1 as follows.

Axiom B. 3 (Strong Negative Dependence) $\succsim$ *satisfies Axiom B.1 and, if the first relation in (4) is strict, the second relation too is strict for some* $\left(x_o, (x_i)_{i\in I}\right) \in \mathcal{X}$ *and* $j \notin I$.

This axiom thus requires that the agent be "sufficiently sensible to externalities."

Proposition 3 *Let* $\succsim$ *on* $\mathcal{F}$ *be a binary relation that satisfies Axioms A.1-A.6. The following statements are equivalent:*

(i) $\succsim$ *satisfies Axioms A.7-A.10 and B.1;*

(ii) $\succsim$ *satisfies Axiom B.3;*

(iii) $\succsim$ *coincides with* $\overset{\cdot}{\succsim}$ *on C.*

Remark 2 If $\succsim$ is represented as in Theorem 1, then Axiom B.3 is clearly satisfied whenever ϱ is strictly increasing in the second component (w.r.t. stochastic dominance). On the contrary, if $\varrho \equiv 0$ we are in the standard expected utility case: Axiom B.1 is satisfied, while Axiom B.3 is violated.

As already observed, Axiom B.1 guarantees that $\overset{\cdot}{\succsim}$ coarser than $\succsim$. Next example shows that this can happen in nontrivial ways.

Example 1 Assume $|S| = |N| = 1$ and $C = \mathbb{R}$, and consider the preferences on $\mathcal{F}$ represented by

$$V(x_o) = x_o,$$

$$V(x_o, x_{-o}) = x_o + \left((x_o)^+ - (x_{-o})^+ \right)^{1/3},$$

for all $x_o, x_{-o} \in \mathbb{R}$. They have a natural interpretation: there is a "poverty line" at 0 and agents do not care about peers below that line. Using Theorem 1, it is easy to check that these preferences satisfy Axioms A.1-A.6 and B.1-B.2. Moreover, it is easy to check that Axiom B.3 is violated. In fact, $\succsim$ coincides on $\mathbb{R}$ with the usual order, while $\overset{\cdot}{\succsim}$ is trivial on $\mathbb{R}_-$ and is the usual order on $\mathbb{R}_+$ (Proposition 3 implies that Axiom B.3 is violated).

6   Special Cases

We consider three special cases of Theorem 2. In the first one only the peers' average social payoff matters, as often assumed in empirical work. In the second one, only the peers' maximum and minimum social payoffs matter. In the last special case we consider separately the peers who have socially higher payoffs than the decision maker o from those that, instead, have worse payoffs. In this way we can separate *invidia* (envy) effects from *superbia* (pride) ones.

6.1   Average Payoff

Let n be a positive integer and $\left(x_o, (x_i)_{i \in I} \right)$ an element of $\mathcal{X}$. Intuitively, an *n-replica* of $\left(x_o, (x_i)_{i \in I} \right)$ is a society in which each agent i in I has spawned $n - 1$ clones of himself, each with the same endowment x_i.

We denote such replica by $\left(x_o, n\,(x_i)_{i \in I} \right)$, which therefore corresponds to an element

$$\left(x_o, \left(x_{i_{J_i}} \right)_{i \in I} \right) \in \mathcal{X},$$

where $\{ J_i \}_{i \in I}$ is a class of disjoint subsets of N with $|J_i| = n$ for all $i \in I$.[11]

Axiom A. 12 (Replica Independence) *Let* $\left(x_o, (x_i)_{i \in I} \right), \left(y_o, (y_i)_{i \in I} \right) \in \mathcal{X}$. *Then*

$$\left(x_o, (x_i)_{i \in I} \right) \succsim \left(y_o, (y_i)_{i \in I} \right) \implies \left(x_o, n\,(x_i)_{i \in I} \right) \succsim \left(y_o, n\,(y_i)_{i \in I} \right), \qquad \forall n \in \mathbb{N}.$$

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[11]Remember that $x_{i_{J_i}}$ is the constant vector taking value x_i on each element of J_i. Notice also that, if $I = \emptyset$, then $\left(x_o, n\,(x_i)_{i \in I} \right) = (x_o) = \left(x_o, (x_i)_{i \in I} \right)$, if $n\,|I| > |N|$, then $\left(x_o, (x_i)_{i \in I} \right)$ admits no n-replicas.

Axiom A. 13 (Randomization Independence) *Let* $\left(x_o, (x_i)_{i \in I}\right), \left(x_o, (y_i)_{i \in I}\right) \in \mathcal{X}$. *If*

$$\left(x_o, (\alpha x_i + (1-\alpha) w_i)_{i \in I}\right) \succ \left(x_o, (\alpha y_i + (1-\alpha) w_i)_{i \in I}\right)$$

for some α *in* $(0,1]$ *and* $\left(x_o, (w_i)_{i \in I}\right) \in \mathcal{X}$, *then*

$$\left(x_o, (\beta x_i + (1-\beta) z_i)_{i \in I}\right) \succsim \left(x_o, (\beta y_i + (1-\beta) z_i)_{i \in I}\right)$$

for all β *in* $(0,1]$ *and* $\left(x_o, (z_i)_{i \in I}\right) \in \mathcal{X}$.

Axioms A.12 and A.13 say, respectively, that the agent's preferences are not reversed either by an n-replica of the societies $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ or by a randomization with a common society $(w_i)_{i \in I}$.

Next we have a standard continuity axiom.

Axiom A. 14 (Continuity) *For all* $\left(x_o, (x_i)_{i \in I}\right), \left(x_o, (y_i)_{i \in I}\right), \left(x_o, (w_i)_{i \in I}\right) \in \mathcal{X}$, *the sets*

$$\left\{\alpha \in [0,1] : \left(x_o, (\alpha x_i + (1-\alpha) w_i)_{i \in I}\right) \succsim \left(x_o, (y_i)_{i \in I}\right)\right\}, \quad \text{and}$$
$$\left\{\alpha \in [0,1] : \left(x_o, (\alpha x_i + (1-\alpha) w_i)_{i \in I}\right) \precsim \left(x_o, (y_i)_{i \in I}\right)\right\},$$

are closed.

To state our result we need some notation. The natural version of diago-nullity for a function ϱ on $K \times (K \cup \{\infty\})$ requires that $\varrho(z, z) = 0 = \varrho(z, \infty)$ for all $z \in K$.[12] Moreover, a function $\varphi : K \to \mathbb{R}$ is *continuously decreasing* if it is a strictly increasing transformation of a continuous and decreasing function $\psi : K \to \mathbb{R}$.[13]

If we add Axioms A.12-A.14 to those in Theorem 2, then we obtain the following representation:

Theorem 3 *Let N be infinite. A binary relation $\succsim$ on $\mathcal{F}$ satisfies Axioms A.1-A.14 if and only if there exist two non-constant affine functions $u, v : C \to \mathbb{R}$, a diago-null function $\varrho : v(C) \times (v(C) \cup \{\infty\}) \to \mathbb{R}$ increasing in the first component and continuously decreasing in the second one on $v(C)$, and a probability P on Σ such that v represents $\overset{\cdot}{\succsim}$ and*

$$V\left(f_o, (f_i)_{i \in I}\right) = \int_S u(f_o(s)) \, dP(s) + \int_S \varrho\left(v(f_o(s)), \frac{1}{|I|} \sum_{i \in I} v(f_i(s))\right) dP(s) \qquad (11)$$

represents $\succsim$ and satisfies $V(\mathcal{F}) = u(C)$.

In the representation (11) decision makers only care about the average social value $|I|^{-1} \sum_{i \in I} v(f_i(s))$. For example, if $v(x) = x$, then (11) becomes

$$V\left(f_o, (f_i)_{i \in I}\right) = \int_S u(f_o(s)) \, dP(s) + \int_S \varrho\left(f_o(s), \frac{1}{|I|} \sum_{i \in I} f_i(s)\right) dP(s),$$

where only the average outcome appears, as it is the case in many specifications used in applications. It is also possible to give behavioral conditions, omitted for brevity, such that $\varrho(z, t) = \gamma(z - t)$ for some increasing $\gamma : \mathbb{R} \to \mathbb{R}$ with $\gamma(0) = 0$.

Finally, as to uniqueness, we have:

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[12]Here K is a nontrivial interval and we adopt the convention $0/0 = \infty$.

[13]For example, strictly decreasing functions φ $(= \bar{\varphi} \circ (-\text{id})$ where $\bar{\varphi}(t) = \varphi(-t)$ for all $t)$ and continuous decreasing functions φ $(= \text{id} \circ \varphi)$ are clearly continuously decreasing, while decreasing step functions are not (unless they are constant).

Proposition 4 *Two quadruples* (u, v, ϱ, P) *and* $(\hat{u}, \hat{v}, \hat{\varrho}, \hat{P})$ *represent the same relations* $\succsim$ *and* $\overset{\cdot}{\succsim}$ *as in Theorem 3 if and only if* $\hat{P} = P$ *and there exist* $\alpha, \beta, \dot{\alpha}, \dot{\beta} \in \mathbb{R}$ *with* $\alpha, \dot{\alpha} > 0$ *such that* $\hat{u} = \alpha u + \beta$, $\hat{v} = \dot{\alpha} v + \dot{\beta}$, *and*

$$\hat{\varrho}(z, r) = \alpha \varrho \left(\frac{z - \dot{\beta}}{\dot{\alpha}}, \frac{r - \dot{\beta}}{\dot{\alpha}} \right)$$

for all $(z, r) \in \hat{v}(C) \times (\hat{v}(C) \cup \{\infty\})$.

6.2 Maximum and Minimum Payoffs

In some cases, agents might only care about the best and worst outcomes that their peers get, rather than the entire distribution of outcomes. This property is captured by the next axiom.

Axiom C. 1 (Best-Worst) *Let* $\left(x_o, (x_i)_{i \in I} \right), \left(x_o, (y_j)_{j \in J} \right) \in \mathcal{X}$. *If*

(i) *for all* $i \in I$ *there is* $j \in J$ *such that* $y_j \overset{\cdot}{\succsim} x_i$,

(ii) *for all* $j \in J$ *there is* $i \in I$ *such that* $y_j \overset{\cdot}{\succsim} x_i$,

then $\left(x_o, (x_i)_{i \in I} \right) \succsim \left(x_o, (y_j)_{j \in J} \right)$.

The intuition here is that, for a fixed "level of consumption" of o, in society $(y_j)_{j \in J}$ is more difficult to *keep up with the Joneses* (point (i)), while in society $(x_i)_{i \in I}$ is easier to *stay ahead of them* (point (ii)).

Given a non-singleton interval $K \subseteq \mathbb{R}$, set

$$K^{1,2} = \{(z, r, t) \in K \times K \times K : r \leq t\} \cup (K \times \{+\infty\} \times \{-\infty\}).$$

Diago-nullity on $K^{1,2}$ takes the form $\varrho(z, z, z) = 0 = \varrho(z, +\infty, -\infty)$ for all $z \in K$, with the conventions $\min \emptyset = +\infty$ and $\max \emptyset = -\infty$.

Theorem 4 *Let* $|N| > 1$. *A binary relation* $\succsim$ *on* $\mathcal{F}$ *satisfies Axioms A.1-A.11 and C.1 if and only if there exist two non-constant affine functions* $u, v : C \to \mathbb{R}$, *a diago-null function* $\varrho : v(C)^{1,2} \to \mathbb{R}$, *increasing in the first component and decreasing in the second and third ones on* $v(C)$, *and a probability* P *on* Σ *such that* v *represents* $\overset{\cdot}{\succsim}$ *and*

$$V \left(f_o, (f_i)_{i \in I} \right) = \int_S u \left(f_o(s) \right) dP(s) \tag{12}$$
$$+ \int_S \varrho \left(v(f_o(s)), \min_{i \in I} v(f_i(s)), \max_{i \in I} v(f_i(s)) \right) dP(s)$$

represents $\succsim$ *and satisfies* $V(\mathcal{F}) = u(C)$.

In the representation (12) only the best and worst values $\min_{i \in I} v(f_i(s))$ and $\max_{i \in I} v(f_i(s))$ are relevant for the decision maker o. Here uniqueness takes the following form:

Proposition 5 *Two quadruples* (u, v, ϱ, P) *and* $(\hat{u}, \hat{v}, \hat{\varrho}, \hat{P})$ *represent the same relations* $\succsim$ *and* $\overset{\cdot}{\succsim}$ *as in Theorem 4 if and only if* $\hat{P} = P$ *and there exist* $\alpha, \beta, \dot{\alpha}, \dot{\beta} \in \mathbb{R}$ *with* $\alpha, \dot{\alpha} > 0$ *such that* $\hat{u} = \alpha u + \beta$, $\hat{v} = \dot{\alpha} v + \dot{\beta}$, *and*

$$\hat{\varrho}(z, r, R) = \alpha \varrho \left(\frac{z - \dot{\beta}}{\dot{\alpha}}, \frac{r - \dot{\beta}}{\dot{\alpha}}, \frac{R - \dot{\beta}}{\dot{\alpha}} \right)$$

for all $(z, r, R) \in \hat{v}(C)^{1,2}$.

6.3 Separating Invidia and Superbia

The previous special case suggests to consider an agent who, in a given reference society, considers separately those who socially dominate him and those who are socially dominated.[14] Next Axiom D.1 describes an agent whose welfare decreases when a dominant element joins the society (invidia), while it increases if a dominated element joins it (superbia).

Axiom D. 1 Let $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$, $j \notin I$, and $c \in C$.

(i) If $c \overset{\cdot}{\succsim} x_o$, then $\left(x_o, (x_i)_{i \in I}\right) \succsim \left(x_o, (x_i)_{i \in I}, c_{\{j\}}\right)$

(ii) If $c \overset{\cdot}{\prec} x_o$, then $\left(x_o, (x_i)_{i \in I}\right) \precsim \left(x_o, (x_i)_{i \in I}, c_{\{j\}}\right)$.

If K is an interval of real numbers, we denote by $\text{pid}\,(K)$ the set of triplets (z, μ, μ') such that $z \in K$, μ and μ' are positive integer measures finitely supported in $K \cap (-\infty, z)$ and $K \cap [z, \infty)$, with $(\mu + \mu')(K) \leq |N|$. The elements of $\text{pid}\,(K)$ are understood to be (payoff to o, distribution of dominated payoffs, distribution of dominant payoffs) triplets. The natural version of the definition of diago-nullity for a function ϱ defined on $\text{pid}\,(K)$ requires that $\varrho(z, 0, n\delta_z) = 0$ for all $z \in K$ and $0 \leq n \leq |N|$.

Theorem 5 A binary relation $\succsim$ on $\mathcal{F}$ satisfies Axioms A.1-A.11 and D.1 if and only if there exist two non-constant affine functions $u, v : C \to \mathbb{R}$, a diago-null function $\varrho : \text{pid}\,(v\,(C)) \to \mathbb{R}$ increasing in the first component, decreasing in the second and third components (w.r.t. lower dominance and upper dominance, respectively), and a probability P on Σ such that v represents $\overset{\cdot}{\succsim}$ and

$$V\left(f_o, (f_i)_{i \in I}\right) = \int_S u\left(f_o\left(s\right)\right) dP\left(s\right) \tag{13}$$

$$+ \int_S \varrho\left(v\left(f_o\left(s\right)\right), \sum_{i : v(f_i(s)) < v(f_o(s))} \delta_{v(f_i(s))}, \sum_{i : v(f_i(s)) \geq v(f_o(s))} \delta_{v(f_i(s))}\right) dP\left(s\right)$$

represents $\succsim$ on $\mathcal{F}$ and satisfies $V\left(\mathcal{F}\right) = u\left(C\right)$.

In this final special case we separate the peers' that have socially higher payoffs $v\left(f_i\left(s\right)\right) \geq v\left(f_o\left(s\right)\right)$ from those that have worst ones $v\left(f_i\left(s\right)\right) < v\left(f_o\left(s\right)\right)$. Suitable specifications of ϱ then model different attitudes of the decision makers toward the former and the latter payoffs.

Proposition 6 Two quadruples (u, v, ϱ, P) and $(\hat{u}, \hat{v}, \hat{\varrho}, \hat{P})$ represent the same relations $\succsim$ and $\overset{\cdot}{\succsim}$ as in Theorem 5 if and only if $\hat{P} = P$ and there exist $\alpha, \beta, \dot{\alpha}, \dot{\beta} \in \mathbb{R}$ with $\alpha, \dot{\alpha} > 0$ such that $\hat{u} = \alpha u + \beta$, $\hat{v} = \dot{\alpha} v + \dot{\beta}$, and

$$\hat{\varrho}\left(z, \sum_{i \in I} \delta_{z_i}, \sum_{j \in J} \delta_{r_j}\right) = \alpha \varrho\left(\dot{\alpha}^{-1}\left(z - \dot{\beta}\right), \sum_{i \in I} \delta_{\dot{\alpha}^{-1}(z_i - \dot{\beta})}, \sum_{j \in J} \delta_{\dot{\alpha}^{-1}(r_j - \dot{\beta})}\right)$$

for all $\left(z, \sum_{i \in I} \delta_{z_i}, \sum_{j \in J} \delta_{r_j}\right) \in \text{pid}\,(\hat{v}\,(C))$.

As a final remark, observe that if in Axiom D.1 $c \overset{\cdot}{\succsim} x_o$ and $c \overset{\cdot}{\prec} x_o$ are replaced by, respectively, $c \overset{\cdot}{\succ} x_o$ and $c \overset{\cdot}{\precsim} x_o$, then the representation (13) still holds provided $\sum_{i : v(f_i(s)) < v(f_o(s))} \delta_{v(f_i(s))}$ and $\sum_{i : v(f_i(s)) \geq v(f_o(s))} \delta_{v(f_i(s))}$ are replaced by $\sum_{i : v(f_i(s)) \leq v(f_o(s))} \delta_{v(f_i(s))}$ and $\sum_{i : v(f_i(s)) > v(f_o(s))} \delta_{v(f_i(s))}$, respectively. Relative to the original version, this variation of Axiom D.1 captures different attitudes towards the addition of an agent with the "same" outcome as o.

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[14] Even if formally this happens only if $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$ is such that $\min_{i \in I} v\left(x_i\right) \leq v\left(x_o\right) \leq \max_{i \in I} v\left(x_i\right)$.

7 Behavioral Attitudes

The axiomatization of preferences given in the first two basic theorems opens now the way to a behavioral foundation of the analysis of preferences. In this section we assume that $\succsim$ satisfies Axioms A.1-A.14, so that the representation (11) holds.

It is sometimes useful to consider attitudes that hold "locally," on subsets of C. For this reason, throughout this section we denote by D a convex subset of C. An event $E \in \Sigma$ is *ethically neutral* if $\bar{c}E\underline{c} \sim \underline{c}E\bar{c}$ for some $\bar{c} \nsim \underline{c}$ in C, representation (11) guarantees that this amounts to say that the agent assigns probability $1/2$ to event E.

7.1 Social Loss Aversion

An outcome profile where your peers get a socially better outcome than yours can be viewed as social loss; conversely, a profile where you get more than them can be viewed as a social gain. This taxonomy is important because individuals might well have different attitudes toward such social gains and losses, similarly to what happens for standard private gains and losses.

We say that a preference $\succsim$ is *more envious than proud* (or *averse to social losses*), relative to an ethically neutral event E, a convex set $D \subseteq C$, and a given $x_o \in D$, if

$$(x_o, x_o) \succsim (x_o, x_i E y_i) \tag{14}$$

for all $x_i, y_i \in D$ such that $(1/2)\,x_i + (1/2)\,y_i \overset{\cdot}{\sim} x_o$. The intuition is that agent o tends to be more frustrated by envy than satisfied by pride (or, assuming w.l.o.g. $x_i \overset{\cdot}{\succsim} y_i$, he is more scared by the social loss (x_o, x_i) than lured by the social gain (x_o, y_i)).

Proposition 7 *If $\succsim$ admits a representation (11), then $\succsim$ is more envious than proud, relative to an ethically neutral event E, a convex $D \subseteq C$, and $x_o \in D$ if and only if*

$$\varrho(v\,(x_o)\,, v\,(x_o) + h) \le -\varrho(v\,(x_o)\,, v\,(x_o) - h) \tag{15}$$

for all $h \ge 0$ such that $v\,(x_o) \pm h \in v\,(D)$. In particular,[15]

$$D_{+}\varrho\,(v\,(x_o)\,, v\,(x_o)) \le D_{-}\varrho\,(v\,(x_o)\,, v\,(x_o))\,. \tag{16}$$

provided $v\,(x_o) \in \operatorname{int}\,(v\,(D))$.

An immediate implication of Proposition 7 is that, given D and x_o, $\succsim$ is more envious than proud relatively to an ethically neutral event E if and only if it is more envious than proud relatively to any other ethically neutral event. In other words, the choice of E is immaterial in the definition of social loss aversion.

7.2 Social Risk Aversion

More generally, decision makers may dislike uncertainty about their peers' social standing. This suggests to strenghten the notion we just discussed as follows. Say that a preference $\succsim$ is *averse to social risk*, relatively to an ethically neutral event E, a convex set $D \subseteq C$, and a given $x_o \in C$, if

$$(x_o, w_i) \succsim (x_o, x_i E y_i) \tag{17}$$

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[15]Here $D_{+}\varrho\,(r, r) = \liminf_{h \downarrow 0} \frac{\varrho(r, r+h) - \varrho(r,r)}{h}$ and $D_{-}\varrho\,(r, r) = \liminf_{h \uparrow 0} \frac{\varrho(r, r+h) - \varrho(r,r)}{h}$.

for all $x_i, y_i, w_i \in D$ such that $(1/2) x_i + (1/2) y_i \stackrel{\cdot}{\sim} w_i$. Notice that the previous definition of being more envious than proud requires that (17) holds only for $w_i = x_o$.[16]

The next result characterizes social risk aversion in terms of concavity of ϱ.

Proposition 8 *If $\succsim$ admits a representation (11), then $\succsim$ is averse to social risk, relative to an ethically neutral event E, a convex $D \subseteq C$, and $x_o \in C$ if and only if $\varrho(v(x_o), \cdot)$ is concave on $v(D)$.*

Propensity to social risk is defined analogously, and characterized by convexity of $\varrho(v(x_o), \cdot)$ on $v(D)$. More importantly, the standard analysis of risk attitudes applies to our more general "social" setting: for example, coefficients of social risk aversion can be studied and compared.

Similarly to what happened for social loss aversion, also here it is immediate to see that the choice of E in the definition of social risk aversion is immaterial.

8   Comparative Interdependence

Next we show that in the very general context of preferences represented as in Lemma 1, and in particular, in all the special cases we considered so far, comparative attitudes are determined by the externality function ϱ. In this section we consider two preferences $\succsim_1$ and $\succsim_2$ on $\mathcal{F}$ both satisfying A.1-A.6, and we denote by $u_n : C \to \mathbb{R}$ and $\varrho_n : \mathrm{pim}(C) \to \mathbb{R}$ the two functions representing $\succsim_n$ in the sense of the basic representation (3) for $n = 1, 2$.

8.1   Aversion to Social Ranking

A decision maker is more averse to social ranking than another one if he has more to lose (in subjective terms) from social comparisons. Formally, say that $\succsim_1$ *more ranking averse than* $\succsim_2$ if

$$\left(x_o, (x_i)_{i \in I} \right) \succsim_1 c_{I_o} \implies \left(x_o, (x_i)_{i \in I} \right) \succsim_2 c_{I_o}, \tag{18}$$

for all $\left(x_o, (x_i)_{i \in I} \right) \in \mathcal{X}$ and $c \in C$. In other words, $\succsim_1$ is more ranking averse than $\succsim_2$ if, whenever $\succsim_1$ prefers a possibly unequal social profile to an egalitarian one, then the same is true for $\succsim_2$.[17]

Proposition 9 *Given two preferences $\succsim_1$ and $\succsim_2$ on $\mathcal{F}$ that satisfy Axioms A.1-A.6, the following conditions are equivalent:*[18]

(i) $\succsim_1$ is more ranking averse than $\succsim_2$,

(ii) $u_1 \approx u_2$ and (provided $u_1 = u_2$) $\varrho_1 \leq \varrho_2$.

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[16]A more general definition of social risk aversion can be actually given, without requiring that E is ethically neutral, but just essential. See Appendix 11.4 for details.

[17]*Mutatis mutandis*, the use of egalitarian profiles as benchmarks for equality makes this comparative notion similar in spirit to how Ghirardato and Marinacci (2002) define comparative ambiguiy aversion by using constant acts as benchmark unambiguous acts.

[18]Recall that $u_1 \approx u_2$ means that there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u_1 = \alpha u_2 + \beta$.

A Closer Look Let us have a closer look at ranking aversion. First observe that, by the first part of (ii) of Proposition 9, if two preferences $\succsim_1$ and $\succsim_2$ can be ordered by ranking aversion, then they are *outcome equivalent*; that is, they agree on the set C (precisely, on $\{(c) : c \in C\}$).

If we consider the preferences on the set of all outcome profiles, we can then see that comparability according to ranking aversion can be decomposed in two components:

(a) $x_o \succ_2 y_o \succsim_2 \left(x_o, (x_i)_{i \in I}\right)$ implies $x_o \succ_1 y_o \succsim_1 \left(x_o, (x_i)_{i \in I}\right)$, and

(b) $\left(x_o, (x_i)_{i \in I}\right) \succsim_1 y_o \succ_1 x_o$ implies $\left(x_o, (x_i)_{i \in I}\right) \succsim_2 y_o \succ_2 x_o$.

Condition (a) says that, if a society $(x_i)_{i \in I}$ makes the decision maker 2 dissatisfied of his outcome x_o, then it makes 1 dissatisfied too. In this case we say that $\succsim_1$ is *more envious than* $\succsim_2$.

Similarly, (b) means that every time the decision maker 1 prefers to have the intrinsically inferior outcome x_o in a society $(x_i)_{i \in I}$ than the superior y_o in solitude (or in an egalitarian society), then the same is true for 2. In this case we say that $\succsim_1$ is *less proud than* $\succsim_2$.

The next result shows how ranking aversion can be expressed in terms of the two behavioral traits we just described.

Proposition 10 *Given two preferences $\succsim_1$ and $\succsim_2$ on $\mathcal{F}$ that satisfy Axioms A.1-A.6, the following conditions are equivalent:*

(i) *$\succsim_1$ is more ranking averse than $\succsim_2$,*

(ii) *$\succsim_1$ is outcome equivalent to $\succsim_2$, more envious, and less proud.*

8.2 Social Sensitivity

Decision makers are more socially sensitive when they have more at stake, in subjective terms, from social comparisons; intuitively, they are at the same time more envious and more proud.[19]

Next we show that this notion of social sensitivity is characterized in the representation through a ranking of the absolute values of ϱ.

Proposition 11 *Given two preferences $\succsim_1$ and $\succsim_2$ on $\mathcal{F}$ that satisfy Axioms A.1-A.6, the following conditions are equivalent:*

(i) *$\succsim_1$ is outcome equivalent to $\succsim_2$, more envious, and more proud,*

(ii) *$u_1 \approx u_2$ and (provided $u_1 = u_2$) $|\varrho_1| \geq |\varrho_2|$ and $\varrho_1 \varrho_2 \geq 0$.*

9 Social Economics

We now investigate some economic consequences of our axiomatic analysis. We first introduce a two-period storage economy where agents have our social preferences. We then specialize this economy in order to focus on two distinct important economic phenomena that arise with our preferences, that is, overconsumption/workaholism and conformism/anticonformism.

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[19]$\succsim_1$ is more proud than $\succsim_2$ when the implication in the above point (b) is reversed, i.e. $\left(x_o, (x_i)_{i \in I}\right) \succsim_2 y_o \succ_2 x_o$ implies $\left(x_o, (x_i)_{i \in I}\right) \succsim_1 y_o \succ_1 x_o$.

We consider economies with a continuum of individually negligible agents, for two main reasons: it simplifies an already complicated derivation and it allows to abstract from strategic interactions among agents that might otherwise arise, so that we can better focus on the interdependencies due to the social dimension of preferences.

Formally, the set I of agents is a complete nonatomic probability space (I, Λ, λ). In particular, we denote by M^n the collection of all Λ-measurable functions $\phi : I \to \mathbb{R}^n$ and by L^n the subset of M^n consisting of bounded functions.

9.1 A Storage Economy

There is a single consumption good, which can be either consumed or saved. We consider a *storage economy*, in which a storage technology is available that allows agents to store for their own future consumption any amount of the consumption good they do not consume in the first period.

As we will see momentarily, in the storage economy there is no room for trade: each agent produces, consumes, and saves/stores for his own future consumption. There are no markets and prices, and, with conventional asocial objective functions, this economy is in equilibrium (Definition 2) when agents just solve their individual intertemporal problems (19).

As a result, it is an equilibrium notion limited in scope, with no need of considering any form of mutual compatibility of agents' choices. If, however, agents have our social objective functions, this is no longer the case. In fact, when agents' own consumption choices are affected by their peers' choices, a link among all such choices naturally emerges. Even without any trading, in this case there is a sensible notion of mutual compatibility of the agents' choices and, therefore, a more interesting equilibrium notion becomes appropriate (Definition 3).

In storage economies, therefore, interaction among agents is only due to the social dimension of consumption. This allows us to study the equilibrium effects of this social dimension in "purity," without other factors intruding into the analysis. This is why we consider these economies. Later, in Subsection 9.4, we will briefly discuss a market economy.

We turn now to the formal model. We assume that the storage technology gives a real (gross) return $R > 0$.

Agents live two periods and in each of them they work and consume; in period one they can also store. In the first period each agent i selects a consumption/effort pair $(c_{i,0}, e_{i,0}) \in \mathbb{R}_+^2$, evaluated by a utility function $u_i : \mathbb{R}_+^2 \to \mathbb{R}$. Effort is transformed in consumption good according to an individual production function $F_{i,0} : \mathbb{R}_+ \to \mathbb{R}_+$.

In the second period there is technological uncertainty, described by a stochastic production function $F_{i,s} : \mathbb{R}_+ \to \mathbb{R}_+$ that depends on a finite space S of states of Nature, endowed with a probability P. With the usual abuse of notation we set $S = \{1, 2, ..., S\}$ and $S_0 = \{0, 1, 2, ..., S\}$, and we write p_s instead of $P(s)$. The production functions $\{F_{i,s}\}_{s \in S_0}$ use a physical capital, whose amount is exogenously fixed in each period and state (capital accumulation is thus not studied here).

In the second period too, each agent i works and consumes. He thus selects in each state s a consumption/effort pair $(c_{i,s}, e_{i,s}) \in \mathbb{R}_+^2$, again evaluated by the same utility function u_i of the first period.

Finally, effort is a limited resource: for each i there is a vector $h_i \in \mathbb{R}_+^{S+1}$ such that $e_{i,s}$ cannot exceed $h_{i,s}$ for all $s \in S_0$.

Summing up, the intertemporal problem of agent i in the storage economy is:

$$\max_{(c_i, e_i) \in B_i} U_i(c_i, e_i), \tag{19}$$

where

$$U_i(c_i, e_i) \equiv u_i(c_{i,0}, e_{i,0}) + \beta \sum_{s \in S} p_s u_i(c_{i,s}, e_{i,s}), \qquad \forall (c_i, e_i) \in \mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1},$$

and B_i is the subset of $\mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1}$ consisting of all (c_i, e_i) such that:

(i) $(c_i, e_i) \in \mathbb{R}_+^{S+1} \times \prod_{s=0}^{S} [0, h_{i,s}]$;

(ii) $c_{i,0} \leq F_{i,0}(e_{i,0})$;

(iii) $c_{i,s} = F_{i,s}(e_{i,s}) + R(F_{i,0}(e_{i,0}) - c_{i,0})$ for all $s \in S$.

Since in every period and state each agent can consume all he produces, then B_i is never empty.

We make a first assumption on the storage economy.

H.1 For each agent $i \in I$:

(i) $u_i : \mathbb{R}_+^2 \to \mathbb{R}$ is continuous.

(ii) $F_{i,s} : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and continuous for all $s \in S_0$.

This assumption guarantees that the (nonempty) set B_i is compact, and that the objective function U_i is continuous. By the Weierstrass Theorem, problem (19) thus admits a solution.

Say that a consumption/effort profile $(c, e) \in M^{S+1} \times M^{S+1}$ is *feasible* if $(c_i, e_i) \in B_i$ for all $i \in I$.

Definition 2 *A feasible consumption/effort profile (c^*, e^*) is an* asocial equilibrium *for the storage economy if*

$$U_i(c_i^*, e_i^*) \geq U_i(c_i, e_i), \qquad \forall (c_i, e_i) \in B_i,$$

for λ-almost all $i \in I$.

As we mentioned before, this equilibrium notion just requires that agents individually solve their problems (19), with no interaction whatsoever among themselves.

We turn now to our social preferences. Assume that the preferences of our agents are represented by the preference functional (11), with $\varrho_i(r, t) = \gamma_i(r - t)$, where $\gamma_i : \mathbb{R} \to \mathbb{R}$ is an increasing function with $\gamma_i(0) = 0$. Given a common social value function $v : \mathbb{R}_+ \to \mathbb{R}$, the social objective function V_i of each agent now depends on the entire profiles of consumption and effort, as follows:

$$V_i(c, e) = U_i(c_i, e_i) + \gamma_i\left(v(c_{i,0}) - \int_I v(c_0)\, d\lambda\right) + \beta \sum_{s \in S} p_s \left[\gamma_i\left(v(c_{i,s}) - \int_I v(c_s)\, d\lambda\right)\right],$$

for all $(c, e) \in L_+^{S+1} \times L_+^{S+1}$.[20] Here we are assuming that only consumption has a social dimension, while effort (and leisure) is only valued privately. This was a classic assumption

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[20]Next assumption H.2.i guarantees that all feasible consumption effort/profiles are bounded.

in Veblen's analysis, and is justified by the lower degree of observability of effort relative to consumption. For this reason, effort is not an argument of the function γ.[21]

The equilibrium notion relevant for our social preferences is a Nash equilibrium for a continuum of agents.

Definition 3 *A feasible consumption/effort profile* $(c^*, e^*) \in L^{S+1} \times L^{S+1}$ *is a* social equilibrium *for the storage economy if*

$$V_i(c^*, e^*) \geq V_i\left(c_i, c^*_{-i}, e_i, e^*_{-i}\right), \qquad \forall\,(c_i, e_i) \in B_i, \tag{20}$$

for λ-*almost all* $i \in I$.

This equilibrium notion requires a mutual compatibility of agents' choices and is thus qualitatively very different from that of Definition 2, a difference entirely due to the social dimension of our preferences.

A key theoretical issue is the existence of social equilibria. To prove this, we need the following mild assumption. Point (i) says that the effort and production capacities are limited, while the other points are standard assumptions.

H.2 The following conditions are satisfied:

    (i) $\sup_{i \in I}\left(F_{i,s}\left(h_{i,s}\right) + h_{i,s}\right) < \infty$ for all $s \in S_0$.

    (ii) $\gamma_i : \mathbb{R} \to \mathbb{R}$ is increasing and continuous, with $\gamma_i(0) = 0$, for all $i \in I$.

    (iii) the real valued functions $u_{(\cdot)}(x, y)$, $h_{(\cdot),s}$, $F_{(\cdot),s}(z)$, and $\gamma_{(\cdot)}(t)$ are Λ-measurable on I for each fixed $(x, y, z, t) \in \mathbb{R}^3_+ \times \mathbb{R}$ and $s \in S_0$.

    (iv) $v : \mathbb{R}_+ \to \mathbb{R}$ is increasing and continuous.

We can now prove a general existence result for storage economies. The proof is based on Schmeidler (1973) and Balder (1995).

Theorem 6 *In a storage economy satisfying assumptions H.1 and H.2 there exists a social equilibrium.*

9.2   Consumerism: Overconsumption and Workaholism

The first phenomenon we consider is how overconsumption and workaholism can arise in a social equilibrium. This is an often mentioned behavioral consequence of concerns for relative consumption and here Proposition 12 shows how it emerges in our general analysis.[22]

We focus on a single period version of the storage economy. In fact, as pointed out in the Introduction, trade-offs arise in more general intertemporal settings (i.e., consuming more today leads to lower saving and, possibly, to lower future consumption). The tendency to overconsumption and workaholism that here we identify in the single period setting might be then offset by other forces.

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[21]In Section 10.4 we discuss in some more detail the case in which there are both outcomes that are socially valued (e.g., consumption here) and outcomes that are only privately valued (e.g., effort here).

[22]Empirical evidence on this phenomenon can be found, for example, in the labor economics papers mentioned in the Introduction (see, e.g. Bowles and Park 2005 for a recent study). Recent anecdotal evidence is reported in Rivlin (2007), who describes Silicon Valley workaholic executives as "working class millionaires."

To ease notation, we drop the subscripts 0; that is, c_i and e_i stand for $c_{i,0}$ and $e_{i,0}$, respectively. The asocial problem of each agent $i \in I$ is then given by

$$\max_{(c_i,e_i)\in B_i} u_i\,(c_i, e_i) \qquad (21)$$

where $B_i = \left\{ (c_i, e_i) \in \mathbb{R}_+^2 : 0 \le e_i \le h_i,\ c_i = F_i\,(e_i) \right\}$.

Here a feasible consumption/effort profile $(c^*, e^*) \in L \times L$ is an asocial equilibrium if (c_i^*, e_i^*) is a solution of problem (21) for λ-almost all $i \in I$.

H.3 For each agent $i \in I$:

(i) u_i is twice continuously differentiable on $\mathbb{R}_{++}^2$, $\partial u_i / \partial x > 0$, and the Hessian matrix $\nabla^2 u_i$ is negative definite.

(ii) F_i is twice differentiable on $\mathbb{R}_{++}$, $F_i' > 0$ and $F_i'' < 0$.

Lemma 3 *If H.1, H.2, and H.3 hold, then there exists a (λ-a.e.) unique asocial equilibrium* $(\hat{c}, \hat{e})$.

Social objective functions V_i take the form

$$V_i\,(c, e) = u_i\,(c_i, e_i) + \gamma_i \left(v\,(c_i) - \int_I v\,(c)\,d\lambda \right),$$

and a feasible pair $(c^*, e^*) \in L \times L$ is a social equilibrium if (20) holds.

To state the result we need a condition and some notation. The special form that B_i has in this case guarantees that a consumption/effort profile $(c, e) \in M \times M$ is feasible if and only if $e_i \in [0, h_i]$ and $c_i = F_i\,(e_i)$ for all $i \in I$. Under H.2.i, feasible profiles are thus determined by effort profiles that belong to the supnorm closed and convex set $E = \{e \in L : 0 \le e \le h\}$. The value of the social objective function can be thus written as

$$W_i\,(e) = u_i\,(F_i\,(e_i)\,, e_i) + \gamma_i \left(v\,(F_i\,(e_i)) - \int_I v\,(F_\iota\,(e_\iota))\,d\lambda\,(\iota) \right) \quad \forall e \in E, i \in I. \qquad (22)$$

An equilibrium (c^*, e^*) is *internal* if $e^* \in \mathrm{int}E$ and *strongly Pareto inefficient* if it is strongly Pareto dominated, that is, there is $\varepsilon > 0$ and a feasible $(c, e) \in M \times M$ such that

$$V_i\,(c, e) \ge V_i\,(c^*, e^*) + \varepsilon$$

for λ-almost all $i \in I$.

We can now state the needed assumption.

H.4 The following conditions are satisfied:

(i) v is differentiable on $\mathbb{R}_{++}$ and $v' > 0$.

(ii) γ_i is differentiable on $\mathbb{R}$ for all $i \in I$, and $\inf_{(i,t)\in I\times\mathbb{R}} \gamma_i'\,(t) > 0$.

(iii) $\displaystyle \sup_{|x|,|y|,|t|\le n,\, i\in I} |u_i(x, y) + \gamma_i\,(t)| < \infty$ and $\displaystyle \sup_{|x|\le n,\, i\in I} |v'(x) + F_i'\,(x)| < \infty$

for all $n \in \mathbb{N}$,[23]

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[23]This implies $W\,(E)$ consists of bounded functions.

(iv) $W : E \to L$ is strictly differentiable on int$E$.[24]

Proposition 12 *If H.1-H.4 hold, then internal social equilibria are strongly Pareto inefficient and exhibit overconsumption and workaholism.*[25]

Overconsumption and workaholism thus characterize equilibria in the single period version of the storage economy. We studied here in detail the Pareto inefficiency of the equilibria to stress the negative features of these equilibria.

This result confirms a well known intuition about social preferences. The next section will show a genuine novel economic insight of our analysis.

9.3 Conformism and Anticonformism

We now study how conformism and anticonformism can characterize the consumption choices of agents in social equilibria, depending, as anticipated in the Introduction, on whether either envy or pride prevails among agents.

Since the rise of anticonformism is our main interest, in order to better focus on this issue we consider a version of the storage economy in which agents are *identical* (so that the social dimension of their preferences is the only possible cause of heterogeneous consumption choices). We also assume that labor is supplied inelastically, say $e_{i,s} = \bar{e}_s > 0$ for all $i \in I$ and $s \in S_0$. To ease notation, we set $F_0(\bar{e}_0) = \bar{x}_0 > 0$ and $F_s(\bar{e}_s) = \bar{x}_s > 0$, and we drop effort as argument of the utility function u.

In this case, the asocial intertemporal problem of each identical agent i is

$$\max_{c_i \in [0, \bar{x}_0]} \left(u(c_i) + \beta \sum_{s \in S} p_s u(\bar{x}_s + R(\bar{x}_0 - c_i)) \right). \tag{23}$$

Define $U : [0, \bar{x}_0] \to \mathbb{R}$ by $U(c_i) = u(c_i) + \beta \sum_{s \in S} p_s u(\bar{x}_s + R(\bar{x}_0 - c_i))$ for all $c_i \in [0, \bar{x}_0]$. A (first period) consumption profile $c \in M$ is feasible if it belongs to the set

$$C = \{c \in M : 0 \le c \le \bar{x}_0\},$$

and is an asocial equilibrium if c_i solves problem (23) for λ-almost all $i \in I$. Clearly, all asocial equilibria are symmetric (i.e., constant λ-almost everywhere) provided U is unimodal.

H.5 The following conditions are satisfied:

(i) $u : \mathbb{R}_+ \to \mathbb{R}$ is continuous on $\mathbb{R}_+$, strictly concave, strictly increasing, and differentiable on $(0, +\infty)$.

(ii) $U'_+(0) > 0 > U'_-(\bar{x}_0)$.

Lemma 4 *If H.5 holds, then there exists a (λ-a.e.) unique asocial internal equilibrium.*

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[24]See Clarke [39, pages 30-32] for the properties of strict differentiabilitys.

[25]That is, λ-a.e., $c_i^* > \hat{c}_i$ and $e_i^* > \hat{e}_i$, where (c^*, e^*) and $(\hat{c}, \hat{e})$ are, respectively, social and asocial consumption and effort equilibrium pairs.

Given $\gamma : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R}_+ \to \mathbb{R}$ and $c \in C$, agent i's social objective function becomes

$$V_i(c) = u(c_i) + \beta \sum_{s \in S} p_s u(\bar{x}_s + R(\bar{x}_0 - c_i) + \gamma \left(v(c_i) - \int_I v(c_\iota) \, d\lambda(\iota) \right)$$

$$+ \beta \sum_{s \in S} p_s \left[\gamma \left(v(\bar{x}_s + R(\bar{x}_0 - c_i)) - \int_I v(\bar{x}_s + R(\bar{x}_0 - c_\iota)) \, d\lambda(\iota) \right) \right].$$

Here a $c^* \in C$ is a social equilibrium if $V_i(c^*) \geq V_i\left(c_i, c^*_{-i}\right)$ for all $c_i \in [0, \bar{x}_0]$ and for λ-almost all $i \in I$.

We will use the following assumption.

H.6 The following conditions are satisfied:

   (i) $v : \mathbb{R}_+ \to \mathbb{R}$ is continuous, concave, and strictly increasing on $\mathbb{R}_+$.

   (ii) $\gamma : \mathbb{R} \to \mathbb{R}$ is increasing and $\gamma(0) = 0$.

We can now state the main result of the section. We consider two specular cases, one in which agents exhibit pure envy, that is, $\gamma(t) = 0$ for all $t \geq 0$, and one in which they exhibit pure pride, that is, $\gamma(t) = 0$ for all $t \leq 0$. We show that envy leads to conformism, that is, all social equilibria are symmetric, while pride leads to diversity, that is, all social equilibria are asymmetric.

Theorem 7 *Suppose assumptions H.5 and H.6 hold. Then:*

   (i) *All social equilibria are asymmetric provided $\gamma(t) = 0$ for all $t \leq 0$ and $D_+\gamma(0) > 0$.*

   (ii) *All social equilibria are symmetric provided $\gamma(t) = 0$ for all $t \geq 0$, and $D_+\gamma(t) > 0$ for all $t < 0$.*

Moreover, in case (ii), $D^-\gamma(0) = 0$ implies that the asocial symmetric equilibrium is the unique social equilibrium.

As we remarked in the Introduction, the diversity in consumption behavior caused by pride is the most remarkable part of the result because agents are identical. By point (i) of Theorem 7, in all equilibria necessarily some agents will choose to consume more today , that is, to have a dominant position today, while other agents will choose the opposite, that is, they will save more today in order to consume more tomorrow and have then a dominant position. This diversity in equilibrium behavior is entirely due to the social dimension of preferences.

9.4   A Market Economy: Autarky and Trade

In the storage economy there was no room for trade. A simple modification of the storage economy that allows trade is to change the "saving technology" by assuming that agents no longer can store the consumption good for future consumption. They can, however, borrow and lend amounts of the consumption good, which is now also a real asset. Agents can save by lending any amount of the consumption good that they do not consume in the first period. As a result, in the (real) asset economy agents interact by trading in the real asset market.

Though for brevity we do not study in detail this economy, it is worth observing that here conformism/anticonformism correspond to no trade/trade. In fact, conformism means that all social equilibria are symmetric, and, by the market clearing condition, it is easy to see that in

such equilibria there is no trade in the real asset market. In contrast, this market operates in the asymmetric equilibria of the anticonformism case. As a result, in this market economy, envy leads to autarky, pride to trade.

10 Discussion

10.1 A Regret Interpretation

The theory we have presented is a theory of social emotions like envy and pride. It allows, however, a close comparison with of social emotions private emotions like regret. This is clear if we consider again the representation (1) in a different light.

For this interpretation, consider again f_o as the act chosen by the decision maker. Suppose now that the acts $(f_i)_{i \in I}$ are acts that were available to him, but that he did not choose. When the state s is revealed, the decision maker also knows the outcome of those acts, and can formulate the counterfactual thought of the utility that he would have had he chosen, say a f_i instead of f_o. The representation (1) gives the ex ante value of the vector $(f_o, f_i)_{i \in I}$ as the combination of two components: the expected utility of the chosen act f_o, and the expected regret that will follow from the comparison state by state between the outcome $f_o(s)$ and the vector of outcomes $(f_i(s))_{i \in I}$.[26]

By doing this the decision maker exhibits myopic behavior in that he forgets, in his ex post evaluation of the act f_i at s, the different outcomes that this act would have delivered in different states. In this way a representation is given to a preference order not over societies, but rather over acts profiles: the distinctive act f_o is the chosen one, the others are the acts that were available and not chosen. This interpretation is reasonable only in the case in which $v = u$, that is, for the private utility representation (5).

This connection between envy and regret is particularly useful in experimental work because it provides a way to test one of our main hypothesis, that envy may be the result of the combination of social and private components.

The axioms that we have presented for the representation (5) can be reinterpreted in this regret interpretation of our setting. The interpretation of the first set of axioms A.1-A.4 (Weak Order, Monotonicity, Archimedean, and Independence) is very similar to the one given for the social interpretation, and needs no further comment.

The other axioms need a separate discussion. Conformistic Indifference (axiom A. 5) requires that regret or relief are not possible when all the acts available deliver the same consequence, independently of their number. Anonimity (axiom A. 6) is the requirement that the labeling of non obtained consequences is irrelevant. Negative Dependence (axiom B. 1) is the axiom that makes the analysis a theory of regret: for any given profile of outcomes, if we make one of the unobtained ones better, the resulting outcome profile is less preferred. Comparative Preference (axiom B. 2) requires that the presence of unobtained outcomes stresses the perceived difference in the obtained outcome. For a given gain and loss, a gain facing alternative outcomes is better than a gain without them, and the same for a loss.

The similarity of the interpretation of the axioms in the private and social environment should not hide a deep conceptual difference between the two, which becomes clear when we consider constant acts. In a social environment the preference order between 10 dollars to you and 5 to the other over 10 dollars to you and 15 to the other is clear, and can be made an

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[26]This combination between expected utility and expected regret reminds of the representation in Sarver (2005).

object of choice, for example in an experiment. In a private environment, the comparison is now between 10 dollars obtained and 5 dollars that could have obtained, versus 10 dollars obtained and 15 dollars that could have obtained. A preference order can be inferred with hypothetical questions, but the elicitation by choice is problematic: When the real choice is among constant acts, the subject would never find himself in the second alternative of the choice of 10 dollars obtained when 15 were available. In other words, the regret interpretation is problematic from a revealed preference standpoint (see Section 10.7).

10.2 Related Specifications

To further illustrate our choice criterion (1), we now compare it with other specifications used in the large empirical literature on relative consumption and income effects discussed in the Introduction.

For convenience, consider deterministic monetary profiles $(x_o, (x_i)_{i \in I})$. The following functional form, along with its variations,

$$V\left(x_o, (x_i)_{i \in I}\right) = \frac{1}{1-\gamma} \left(\frac{x_o}{\bar{x}}\right)^{1-\gamma} \tag{24}$$

has been widely used in empirical work, where $\bar{x}$ is the average outcome (say consumption or income).

This functional form is not a special case of our representation (1), which in the deterministic case becomes:

$$V\left(x_o, (x_i)_{i \in I}\right) = u(x_o) + \varrho\left(v(x_o), \sum_{i \in I} \delta_{v(x_i)}\right). \tag{25}$$

In fact, the ratio $x_o/\bar{x}$ captures the externality determined by relative outcome concerns; that is, it plays the same role of the function ϱ in (1). What is altogether missing in the specification (24) is the term $u(x_o)$ of (25), that is, the expected utility $\int_S u(f_o(s)) \, dP(s)$ in the general representation (1).

As a result, the specification (24) and its variations – with, for example, ratios replaced by differences[27] – model decision makers who are only concerned about relative outcome effects, and their own outcomes are only valued in that regard. There is no specific role, instead, for the private emotions determined by the outcomes' intrinsic properties, which in our representation (25) are modelled by the term $u(x_o)$, which is missing in (24). As a result, the specification (24) overlooks the basic trade-off between absolute and relative effects that we discussed in the Introduction and is, therefore, a significant deviation from standard theory (which is not even a special case of (24)).

To see more concretely the exclusive focus of (24) on relative effects, consider a given profile $(x_o, (x_i)_{i \in I})$ and multiply all its terms by a constant $k > 1$. We then have $V\left(kx_o, (kx_i)_{i \in I}\right) = V\left(x_o, (x_i)_{i \in I}\right)$ for the specification (24), even though the decision maker is getting an higher outcome kx_0, possibly much higher since k can be arbitrarily large. The specification (24) only focuses on relative concerns and so it altogether neglects this improvement in the decision maker's own outcome.

In contrast, in our representation (25) we may have $V\left(kx_o, (kx_i)_{i \in I}\right) > V\left(x_o, (x_i)_{i \in I}\right)$. In fact, even if the term ϱ had a ratio form[28] giving the same value to the profiles $(x_o, (x_i)_{i \in I})$ and $(kx_o, (kx_i)_{i \in I})$, we would still have the term $u(x_o)$ which may be such that $u(kx_o) > u(x_o)$.

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[27]For example, in their influential work Campbell and Cochrane (1999) adopt an intertemporal version of (24) with differences in place of ratios.

[28]Say $\varrho\left(v(x_o), \sum_{i \in I} \delta_{v(x_i)}\right) = \gamma(x_o/\bar{x})$ for some strictly increasing function $\gamma : \mathbb{R} \to \mathbb{R}$.

10.3 An Ex Post Perspective

In the choice criterion (1), decision makers anticipate ex ante that they will evaluate, through the term $\varrho\left(v\left(f_o\left(s\right)\right), \sum_{i \in I} \delta_{v\left(f_i\left(s\right)\right)}\right)$, the relative outcome effects once uncertainty resolves and a state s obtains. In other words, they will assess the relative performance of theirs acts f_o with respect to those, f_i, of their peers from the standpoint of the realized state s.

This ex post perspective is essentially due to the separability across states implied by the monotonicity Axiom 2, which is a special instance of the general monotonicity principle that "an alternative is better when it is better in all possible contingencies." The ex post perspective that in our context this, widely adopted, principle entails may be regarded as featuring an ex post, emotional, bias. For, decision makers who rank alternatives in $\mathcal{F}$ according to (1) may be seen as partially overlooking what ex ante was actually available in $\mathcal{F}$, both for them and for their peers. In fact, in $\mathcal{F}$ there might well not be acts' profiles that deliver in each state the ex post most desired result for the decision makers.

We adopted in our analysis Axiom 2 because we believe that it is a behaviorally meaningful axiom and that the consequent possible ex post emotional bias is a key component in how decision makers emotionally react to peers' outcomes. Notice that our decision makers ex ante anticipate this possible ex post emotional bias, and their optimal choices will thus keep it into account.

Thanks to the behavioral nature of our axioms – in particular, of Axiom 2 – the meaningfulness of our modelling choice will be ultimately determined by the experimental performance of our axioms. In any case, the framework we introduced in this paper would be the natural setting where to study possible modifications of this (and other) aspect of our analysis. For example,

$$V\left(f_o, \left(f_i\right)_{i \in I}\right) = \int u\left(f_o\left(s\right)\right) dP\left(s\right) + \varrho\left(\int v\left(f_o\left(s\right)\right) dP\left(s\right), \sum_{i \in I} \delta_{\int v\left(f_i\left(s\right)\right) dP\left(s\right)}\right) \tag{26}$$

is a variation of the choice criterion (1) that does not feature the ex post bias. We expect that a suitable modification of our analysis, in particular of Axiom 2, would be able to deliver (26). A main contribution of this paper is, in fact, to provide a framework where social decision theory issues can be studied.

10.4 Private and Social Outcomes

In our representation (1) the outcomes $\left(f_o\left(s\right), \left(f_i\left(s\right)\right)_{i \in I}\right)$ of the acts have a social dimension and thus enter both terms in the representation. It is possible, however, to expand the representation in order to model choices that may feature both outcomes that have a social dimension and outcomes that do not have it, possibly because of different degrees of observability by peers. For example, in the consumption and saving problem of Section 9 in which agents select optimal consumption and effort pairs, it is plausible to think that consumption is, in general, more easily observed by peers than effort.

Formally, in this case we need to consider outcome spaces that have a Cartesian structure, say $C \times E$, where C is the set of social outcomes and E is that of private ones. To ease notation, consider deterministic profiles $\left(x_o, y_o, \left(x_i\right)_{i \in I}\right) \in C \times E \times C^I$. Here the choices of our decision maker o feature outcome pairs $\left(x_o, y_o\right)$, where x_o is the social outcome (e.g., consumption), while y_o is only private (e.g., effort). The generalized version of the representation (1) is:

$$V\left(x_o, y_o, \left(x_i\right)_{i \in I}\right) = u\left(x_o, y_o\right) + \varrho\left(v\left(x_o\right), \sum_{i \in I} \delta_{v\left(x_i\right)}\right), \tag{27}$$

where the private outcome y_o only enters in the first term. A routine modification of our derivation, omitted for brevity, delivers the extended representation (27). In the economic application of Section 9 we actually use the extended representation (27).

10.5 Intertemporal Version

A natural and important issue is the intertemporal extension of our static choice criterion (1). Though a full analysis of this issue is beyond the scope of this paper, here we study in some detail the deterministic case. We then briefly discuss the more general case with uncertainty.

Consider a deterministic problem with, for convenience, a finite horizon $T \geq 1$. The natural (and standard) way to study this problem is to interpret states as dates. That is, we consider a finite set $S = \{0, 1, 2, ..., T\}$, now viewed as a set of points in time rather than states of Nature. In this case, for all $f \in \mathcal{A}$ and $t \in S$, $f(t)$ represents consumption at date t.

The next assumption is a standard intertemporal separability condition.

Axiom E. 1 *Date 0 is essential,[29] and for any f_o in $\mathcal{A}_o$ and $c, c', \bar{c}, \bar{c}'$ in C, if*

$$
\begin{bmatrix} f_o(\tau) & \text{if } \tau \neq t, t+1 \\ c & \text{if } \tau = t \\ c' & \text{if } \tau = t+1 \end{bmatrix} \succsim \begin{bmatrix} f_o(\tau) & \text{if } \tau \neq t, t+1 \\ \bar{c} & \text{if } \tau = t \\ \bar{c}' & \text{if } \tau = t+1 \end{bmatrix}
$$

holds for some $t < T$, then it holds for every $t < T$.

This axiom, added to Axioms A.1-A.11 delivers a simple discounted extension of our model in the deterministic case.

Theorem 8 *Let $S = \{0, 1, 2, ..., T\}$ with $T \geq 1$ and $\Sigma = 2^S$. A binary relation $\succsim$ on $\mathcal{F}$ satisfies Axioms A.1-A.11, and E.1 if and only if and only if there exist two non-constant affine functions $u, v : C \to \mathbb{R}$, a diago-null function $\varrho : \text{pim}(v(C)) \to \mathbb{R}$ increasing in the first component and decreasing (w.r.t. stochastic dominance) in the second one, and a constant $\beta > 0$ such that v represents $\overset{\cdot}{\succsim}$ and*

$$
V\left(f_o, (f_i)_{i \in I}\right) = \sum_{t=0}^{T} \beta^t \left[u(f_o(t)) + \varrho\left(v(f_o(t)), \sum_{i \in I} \delta_{v(f_i(t))}\right) \right] \tag{28}
$$

for all $\left(f_o, (f_i)_{i \in I}\right) \in \mathcal{F}$, represents $\succsim$ and satisfies $V(\mathcal{F}) = \left(\sum_{t=0}^{T} \beta^t\right) u(C)$.

We close with few remarks on the extension of the criterion (28) to the general case with uncertainty. In particular, a stochastic version of the time separable deterministic criterion (28) is

$$
V_t\left(f_o, (f_i)_{i \in I}\right) = \mathrm{E}^P\left[\sum_{\tau \geq t} \beta^{\tau - t}\left(u(f_o(\tau)) + \varrho\left(v(f_o(\tau)), \sum_{i \in I} \delta_{v(f_i(\tau))}\right)\right) \middle| \Sigma_t \right],
$$

where Σ_t represents the information available to agent o at time t.

An axiomatic derivation of this stochastic criterion is left for future research. Here just notice that the very special case in which $T = 1$, which we used in Section 9, can be easily obtained from Theorem 2 by adding a fictitious state s_0 that corresponds to period 0.

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[29]See page 69 in the appendix.

10.6 Inequity Aversion

Like envy/pride, also inequity aversion is a social emotion that arises in reaction to peers' outcomes. It is based on fairness considerations and we refer the interested reader to Fehr and Schmidt (1999) for a thorough presentation of this emotion. To further show the methodological scope of our analysis, here we briefly show how to model inequity aversion in our general framework.

The starting point are the basic axioms A.1-A.6, along with the basic representation (3) they deliver. The first additional assumption we make is that agent o evaluates peers' outcomes via his own preference.

Axiom F. 1 *Let $\left(x_o, (x_i)_{i \in I}\right), \left(y_o, (y_i)_{i \in I}\right) \in \mathcal{X}$. If $x_i \sim y_i$ for all $i \in I_o$, then $\left(x_o, (x_i)_{i \in I}\right) \sim \left(y_o, (y_i)_{i \in I}\right)$.*

It is easy to see that this axiom is satisfied by preferences that have the private utility representation (5), that is, preferences that satisfy both the basic axioms and Axioms B.1-B.2. The next axiom is, instead, peculiar to inequity aversion. As Fehr and Schmidt (1999, p. 822) write, "... [players] experience inequity if they are worse off in material terms than the other players in the experiment, and they also feel inequity if they are better off." This motivates the following behavioral assumption.

Axiom F. 2 *Let $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$, $j \in I$, and $c \in C$. If either $c \succsim x_j \succsim x_o$ or $x_o \succ x_j \succsim c$, then $\left(x_o, (x_i)_{i \in I}\right) \succsim \left(x_o, (x_i)_{i \in I-\{j\}}, c_{\{j\}}\right)$.*

In other words, agent o dislikes any change in the outcome of a given peer j that in his view increases inequity, either by improving an already better outcome (i.e., $c \succsim x_j \succsim x_o$) or by impairing a worse one (i.e., $x_o \succ x_j \succsim c$).

We can now state the inequity aversion representation result (recall that $\text{pid}(K)$ was defined before Theorem 5).

Theorem 9 *A binary relation $\succsim$ on $\mathcal{F}$ satisfies A.1-A.6 and F.1-F.2 if and only if there exist a non-constant affine function $u : C \to \mathbb{R}$, a diago-null function $\xi : \text{pid}(u(C)) \to \mathbb{R}$ increasing in the second component and decreasing (w.r.t. stochastic dominance) in the third one, and a probability P on Σ such that*

$$V\left(f_o, (f_i)_{i \in I}\right) = \int_S u\left(f_o(s)\right) dP(s) \tag{29}$$

$$+ \int_S \xi\left(u\left(f_o(s)\right), \sum_{i:u(f_i(s))<u(f_o(s))} \delta_{u(f_i(s))}, \sum_{i:u(f_i(s))\geq u(f_o(s))} \delta_{u(f_i(s))}\right) dP(s)$$

represents $\succsim$ on $\mathcal{F}$ and satisfies $V(\mathcal{F}) = u(C)$.

For example, Fehr and Schmidt (1999) assume that there is no uncertainty and that outcomes are monetary; i.e., S is a singleton and $C = \mathbb{R}$. They consider the preference functional

$$V\left(x_o, (x_i)_{i \in I}\right) = x_o - \alpha_o \frac{1}{|I|} \sum_{i \in I} \max\{x_i - x_o, 0\} - \beta_o \frac{1}{|I|} \sum_{i \in I} \max\{x_o - x_i, 0\} \tag{30}$$

where $\beta_o \le \alpha_o$ and $0 \le \beta_o < 1$. This is a special case of (29), where u is the identity and the function ξ is given by

$$\xi\left(z, \sum_{l \in L} \delta_{t_l}, \sum_{j \in J} \delta_{r_j}\right) = \frac{\alpha_o}{|L| + |J|}\left(z\,|J| - \sum_{j \in J} r_j\right) + \frac{\beta_o}{|L| + |J|}\left(\sum_{l \in L} t_l - z\,|L|\right),$$

unless $\left(z, \sum_{l \in L} \delta_{t_l}, \sum_{j \in J} \delta_{r_j}\right) = (z, 0, 0)$, in which case ξ vanishes.

Clearly, ξ is diago-null, increasing in the second argument, and decreasing in the third one (w.r.t. stochastic dominance). This specification of can be obtained from the general representation (29) along the lines of the special cases we discussed in Section 6.

The uniqueness properties of the representation of inequity averse preferences are similar to the ones we obtained so far:

Proposition 13 *Two triplets (u, ξ, P) and $(\hat{u}, \hat{\xi}, \hat{P})$ represent the same relation $\succsim$ as in Theorem 9 if and only if $\hat{P} = P$ and there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that $\hat{u} = \alpha u + \beta$, and*

$$\hat{\xi}\left(z, \sum_{i \in I} \delta_{z_i}, \sum_{j \in J} \delta_{r_j}\right) = \alpha\xi\left((z - \beta)/\alpha, \sum_{i \in I} \delta_{(z_i - \beta)/\alpha}, \sum_{j \in J} \delta_{(r_j - \beta)/\alpha}\right)$$

for all $\left(z, \sum_{i \in I} \delta_{z_i}, \sum_{j \in J} \delta_{r_j}\right) \in \mathrm{pid}\,(\hat{u}\,(C))$.

The Fehr and Schmidt specification (30) is an instance of a more general specification of (29). In fact, consider

$$V\left(x_o, (x_i)_{i \in I}\right) = x_o + \frac{1}{|I|}\sum_{i \in I}\gamma\,(x_o - x_i), \tag{31}$$

where $\gamma : \mathbb{R} \to \mathbb{R}$ is such that $\gamma\,(0) = 0$. This is a special case of (29), with u identity and[30]

$$\xi\left(z, \sum_{l \in L} \delta_{t_l}, \sum_{j \in J} \delta_{r_j}\right) = \frac{1}{|L \cup J|}\left(\sum_{i \in L}\gamma\,(z - t_l) + \sum_{i \in J}\gamma\,(z - r_j)\right).$$

The specification (30) is, in turn, a special case of (31). In fact,

$$\sum_{i \in I}\max\{x_i - x_o, 0\} = \sum_{i \in I: x_i \ge x_o}(x_i - x_o) \quad \text{and} \quad \sum_{i \in I}\max\{x_o - x_i, 0\} = \sum_{i \in I: x_i < x_o}(x_o - x_i)$$

for all $\left(x_o, (x_i)_{i \in I}\right) \in \mathbb{R}^{I_o}$. Then, (31) reduces to (30) by setting

$$\gamma\,(t) = \begin{cases} -\beta_o t & \text{if } t \ge 0, \\ \alpha_o t & \text{if } t < 0, \end{cases}$$

Summing up, we derived an inequity aversion representation that encompasses Fehr and Schmidt (1999) by adding suitable behavioral assumptions to the basic axioms A.1-A.6. The behavioral nature of our derivation allows to use behavioral data to test in a subject the relevance of fairness/inequity considerations, as opposed to, say, envy/pride ones. In fact, it is enough to check experimentally, through choice behavior, whether for example a subject tends to satisfy Axiom B.1 rather than F.2.

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[30]Here $I = L \cup J$ and, since $\left(z, \sum_{l \in L} \delta_{t_l}, \sum_{j \in J} \delta_{r_j}\right) \in \mathrm{pid}\,(u\,(C))$, by definition we have $t_l < z$ for all $l \in L$ and $r_j \ge z$ for all $j \in J$.

Finally, observe that in the representation (30), F.2 is violated and Axiom B.1 is satisfied when $\beta_o < 0 \leq \alpha_o$. In this case (30) becomes a simple and tractable example of the private utility representation (5).[31] This is a possibility mentioned by Fehr and Schmidt (1999), who on p. 824 of their paper observe "... we believe that there are subjects with $\beta_o < 0$ ..." that is, as Veblen (1899, p. 31) wrote long time ago, there are subjects for whom "... it is extremely gratifying to possess something more than others." These subjects experience envy/pride, and so violate axiom F.2 and satisfy B.1.

10.7 Some Methodological Issues

In this subsection we discuss some methodological issues that arise in our analysis.

Social Theory The Veblen critique questions the basic tenet of standard consumer theory that consumers' preferences only depends on the private functions and uses of the consumer goods, that is, on their physical nature, with no role for any possible cultural/symbolic, and so social, aspect they might have. In this way, standard consumer theory can analyze consumers' decisions in isolation, without worrying about any possible externality that such decisions might generate.

In contemporary societies, however, the symbolic value of consumption has come to play a fundamental role in social interactions, much more than ever before in human history. Social scientists often describe contemporary societies as "consumer societies," with consumerism being their distinguishing feature. This is the result of improved living conditions (symbolic consumption has a smaller role in mere subsistence economies), but also of major cultural and technological changes.

For this reason, consumption and its symbolic aspects has been a central research theme in the social sciences, beginning with the seminal works of Barthes (1964) and Baudrillard (1968), (1970) and (1972) in Social Theory, of Sahlins (1976) and Douglas and Isherwood (1979) in Anthropology, and of, in a more applied context, Levy (1959) and Grubb and Grathwohl (1967) in Consumer Research.

At a theoretical level, the most influential works are probably those of Barthes and Baudrillard. Their studies move from Veblen's early analysis of conspicuous consumption, which they gave a theoretical framework by observing that the symbolic aspect of consumer goods makes them a system of signs, a communication system, and, as such, suitable to semiotic analysis.[32] In other words, their theoretical stance is that consumer theory should be viewed as part of Semiotics, to be studied with the concepts and categories elaborated in that area since the seminal works of Saussure. This means, *inter alia*, that consumer theory should be studied with reference to linguistic laws, which in Barthes' "translinguistic" view (see Eco 1976 p. 30) are the reference model for the study of all systems of signs.

However controversial, the works of Barthes and Baudrillard provide an important theoretical perspective on consumption, building on Veblen's original insights. An early application of the semantic approach to consumption is Baudrillard's generalization of Marxian political economy, with the introduction of the "sign value" as a supplement to the classic user and exchange values in order to model the symbolic side of goods (Baudrillard 1972).

This semantic theoretical perspective on consumption gives prominence to the symbolic over

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[31]This happens, more generally, in (31) when γ is an increasing function.

[32]Consumer goods (and, more generally, all objects) are sign-functions in the terminology of Barthes (1964) because, unlike verbal signs, they have a functional origin (see Eco (1976) pp. 48-50).

the functional,[33] and thus emphasizes the social nature of consumption. In fact, the symbolic is, by its nature, social because its meaningfulness relies on the existence of an interpretation code, a "consumer culture," shared by all consumers (Baudrillard 1970). In the large literatures that the works of Barthes and Baudrillard originated,[34] the prominence of the symbolic over the functional became more and more accentuated, with consumption decisions viewed as essentially determined by the symbolic meaning of the goods. In Social Theory, economic exchange is thus reduced to a purely symbolic exchange.

We agree with the Veblenian and social theoretic insight that consumption decisions depend on both the functional and symbolic meanings of the chosen goods. For this reason, these decisions must be studied as social decisions and only a proper modelling of their social dimension can make them meaningful and understandable. This was, in fact, our original motivation and, in a sense, our "social" extension of standard consumer theory parallels that undertaken by Baudrillard for Marxian political economy. We disagree, however, with social theorists in some fundamental methodological and substantive issues.

On the methodological side, this is a paper in theoretical consumer theory: our purpose is to extend, incrementally, standard consumer theory by modelling a social dimension so far overlooked in economic theory.[35] We try to do this in the most parsimonious way, by remaining as close as possible to the standard model. We thus adopt the classic, Weberian, methodological individualism and rational action approach of standard consumer theory, as well as its revealed preference method. Our agents have preferences that, in principle, can be behaviorally elicited and that are represented by objective functions that agents maximize. Unlike standard consumer theory, where the objective function to maximize is a function u that models the intrinsic, material, utility of consumption as determined by its uses, in our general representation (1) the objective function depends both on a conventional function u and on a new function v that models the social dimension of consumption.[36] In Weber (1968)'s terminology, our agents' intentional states have both a private and a social dimension.

The presence of the function u in our representation reflects another major, substantive, difference of our approach relative to Social Theory. In fact, we believe that the functional dimension of consumption still plays an important role in consumer behavior and we do not agree with the social theoretic almost exclusive focus on the symbolic. As Lucretius wrote, "*utilitas expressit nomina rerum.*"[37]

Instrumental Approach   As we have discussed at length, our approach lies within the neo-classical framework and generalizes the standard model by enlarging the scope of agents' preferences. A different route to study some social decision in a neoclassical setting has been pursued by Cole, Mailath, and Postlewaite in a series of influential papers.[38] In a matching model with conventional agents who only care about their own private consumption (and that of their

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[33]Whose role is often regarded as merely ancillary, as an ex post rationalization of a prime symbolic meaning. As Barthes (1964) writes "... once a sign is constituted, society can very well refunctionalize it, and speak about as it were an object made for use: a fur-coat will be described as it served only to protect from the cold." (p. 42 in the English translation).

[34]For reviews, see, e.g., Mick (1986), McCracken (1988), Bocock (1993), and Slater (1997).

[35]Though our preference functional (1) is fully general, application to consumer theory is a main motivation.

[36]In Baudrillard's generalized Marxian political economy, we can view u and v as modelling functional value and symbolic value, respectively.

[37]"Need and use did mould the names of things," *De Rerum Natura*, translation of W. E. Leoanard. We owe this quotation to Rossi-Landi (1968).

[38]See their (1992), (1995), (1998), and (2006) articles, as well as the methodological piece by Postlewaite (1998).

offspring), they elegantly show how status/rank concerns may arise for purely instrumental reasons: an higher status allows better consumption opportunities, which in turn enhance agents' welfare. As Postlewaite (1998) p. 785 said, "individuals have a concern for relative standing because relative standing is instrumental in determining ultimate consumption levels."

The instrumental approach is very appealing and insightful, as Postlewaite (1998) eloquently discusses. There are, however, a few reasons why here we pursue a different approach.

First (and foremost), we believe that envy and pride are fundamental emotions, which shape our attitudes toward competition and dominance, our fundamental social attitudes. Envy and pride are key traits in human behavior, possibly hard-wired over time through evolution (see Robson 2001). Who we are, our "personality," is substantially shaped by these two basic emotions, no less than by the traditional, "utilitarian," emotions of pleasure and pain that agents derive from private, material, consumption and that underlie standard utility functions.[39]

We thus take as primitive the social emotions of envy and pride, on equal footing with the traditional "private" emotions of pleasure and pain. In other words, we do not regard these private emotions as more fundamental for economic modelling. The reason why we believe that it is time to enrich the emotional scope of utility analysis is that in our contemporary societies social emotions play a bigger and bigger role, partly because of major cultural and technological changes, as we argued earlier in this section. This is a basic insight of Social Theory, which we fully agree with. Neglecting these social emotions has been for a long time a reasonable simplifying assumption, certainly at the time when neoclassical utility theory was born. Veblen's analysis was much ahead of his time also because, back then, it was arguably less relevant empirically.[40] But, this is no longer true now, as empirical research is currently showing.[41] To conclude, one should never forget that, like all social sciences, Economics is an historically determined discipline and even its more fundamental assumptions might need an update.

Even if one disagrees with our basic view, there is a different, more pragmatic, reason why it is useful to model social decisions by enlarging preferences' domain. In fact, the objective functions that we derive can be viewed as a reduced form of a more complex, possibly not so easily modelled, setting where social considerations may be properly viewed as instrumental. The large number of empirical papers that incorporate social traits into agents' objective functions is a clear proof of the pragmatic usefulness of our modelling strategy. From this standpoint, the main contribution of our axiomatic analysis is to provide some theoretical discipline and insight on these otherwise ad hoc manipulations of agents' objective functions. As we mentioned before, despite the large empirical literature, very little theoretical work exists on this subject.

Post Hoc, Ergo Propter Hoc   The instrumental view ultimately requires to go beyond emotions, which are just "fictional causes" of behavior, mere *post hoc, ergo propter hoc* explanations, as Skinner remarked in his classic (1953) and (1974) books. The deep causes of behavior have a substantial biological basis, determined over time by evolution.[42] In this vein, several papers

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[39]As well known, Pareto showed that the hedonic interpretation of utility functions is not a theoretical necessity. It is, however, a perfectly legitimate interpretation (see the discussion below on revealed preference) that is still pervasive in Economics, as the widespread reference to marginal utilities shows.

[40]Except, of course, for Veblen's social milieu, which motivated his theory. See Matt (2003) for a description of American consumer society at the time of Veblen.

[41]See in particular the cited works of Dynan, Luttmer, and Ravina, which find evidence of relative consumption concerns, our primary motivation.

[42]See, for example, Gross (2006) for a critique of this "organic" view.

have investigated possible biological/evolutionary explanations of preference patterns.[43] For example, a suitable evolutionary analysis would be in order to model the causes of the envious behavior discussed in Section 1.1.1, that is, learning, dominance, and competition.

The objective of Decision Theory, however, is to model how people choose and how this modelling can be usefully embedded in more general economic models. Emotions can be considered as a first approximation explanation of behavior, sufficient for the Decision Theory purposes, in an utilitarian tradition that traces its origin back at least to Bentham's classic pleasure and pain calculus.

In this regard, observe that there is no conceptual inconsistency, both in standard theory and in its extended version presented here, between adherence to the revealed preference methodology and an emotional/hedonistic interpretation of preference rankings and of the derived objective functions. This was recognized since the very beginning of revealed preference analysis, which "does not preclude the introduction of utility by any who may care to do so, nor will it contradict the results attained by use of related constructs," as Samuelson (1938) p. 62 observed.

Revealed preference analysis is the fundamental Decision Theory methodology that bases, through axiomatic analysis, the derivation of decision makers' objective functions on actual choice behavior. This gives empirical content to utility analysis since, still today, choice behavior is the best observable, and so testable, aspect of human economic behavior.[44]

Whenever possible, it is thus very important to anchor, through axiomatic analysis, objective functions to choice behavior.[45] That said, one can always interpret observed choices as the result of a Benthamian emotional calculus, be either the traditional private pain/pleasure calculus or the more general private and social one advocated here. Interpretation is a semantic exercise and, as such, is legitimate insofar as it provides a reasonable interpretation of the formal model (i.e., of the axioms), whose scientific status, however, is determined by its testable/falsifiable choice foundations.

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[43]See the surveys of Robson (2001) and Rayo and Becker (2007a), as well as the results in social cognitive neuroscience overviewed in Frith and Frith (1999), Blakemore, Winston, and Frith (2004), Gallese, Keysersa, and Rizzolatti (2004), Moll, Zahn, de Oliveira-Souza, Krueger, and Grafman (2005), and Lieberman (2007).

[44]Some recent technical advances, especially in the neurosciences (e.g., neuro scans), might suggest a future where other sources of data will acquire a status comparable to that of choice behavior. In this case, revealed preference analysis will have to expand its scope accordingly.

[45]As Schumpeter (1954) p. 1059 simply put it, "... nobody will deny that it is preferable to derive a given set of propositions from externally or 'objectively' observable facts, if it can be done, than to derive the same set of propositions from premises established by introspection..."

11 Proofs and Related Material

11.1 Distribution Functions

Let $n, m \in \mathbb{N}$, $I = \{i_1, ..., i_n\}$, $J = \{j_1, ..., j_m\}$, $\mathbf{a} = (a_{i_1}, a_{i_2}, ..., a_{i_n}) \in \mathbb{R}^I$, and $\mathbf{b} = (b_{j_1}, ..., b_{j_m}) \in \mathbb{R}^J$. In this subsection, we regroup some useful results on stochastic dominance.

Lemma 5 *If $a_{i_1} \leq a_{i_2} \leq ... \leq a_{i_n}$ and $b_{j_1} \leq b_{j_2} ... \leq b_{j_m}$, then the following facts are equivalent:*

(i) $F_{\mathbf{a}}(t) \leq F_{\mathbf{b}}(t)$ *for all $t \in \mathbb{R}$.*

(ii) $n \leq m$ *and $F_{\mathbf{a}}(t) \leq F_{(b_{j_1}, ..., b_{j_n})}(t)$ for all $t \in \mathbb{R}$.*

(iii) $n \leq m$ *and $a_{i_k} \geq b_{j_k}$ for all $k = 1, ..., n$.*

Lemma 6 *If $a_{i_1} \leq a_{i_2} \leq ... \leq a_{i_n}$ and $b_{j_1} \leq b_{j_2} ... \leq b_{j_m}$, then the following facts are equivalent:*

(i) $G_{\mathbf{a}}(t) \geq G_{\mathbf{b}}(t)$ *for all $t \in \mathbb{R}$.*

(ii) $n \geq m$ *and $G_{(a_{i_{n-m+1}}, a_{i_{n-m+2}}, ..., a_{i_n})}(t) \geq G_{\mathbf{b}}(t)$ for all $t \in \mathbb{R}$.*

(iii) $n \geq m$ *and $a_{i_{n-m+k}} \geq b_{j_k}$ for all $k = 1, ..., m$.*

Lemma 7 *The following statements are equivalent:*

(i) $\mu_{\mathbf{a}}$ *stochastically dominates $\mu_{\mathbf{b}}$.*

(ii) $n = m$ *and if σ and τ are permutations of $\{1, ..., n\}$ such that $a_{i_{\sigma(1)}} \leq a_{i_{\sigma(2)}} \leq ... \leq a_{i_{\sigma(n)}}$ and $b_{j_{\tau(1)}} \leq b_{j_{\tau(2)}} \leq ... \leq b_{j_{\tau(n)}}$, then $a_{i_{\sigma(k)}} \geq b_{j_{\tau(k)}}$ for all $k = 1, ..., n$.*

(iii) $n = m$ *and there exists a permutation ζ of $\{1, ..., n\}$ such that $a_{i_{\zeta(k)}} \geq b_{j_k}$ for all $k = 1, ..., n$.*

(iv) *There exists a bijection $\pi : I \to J$ such that $a_i \geq b_{\pi(i)}$ for all $i \in I$.*

(v) $|I| = |J|$ *and $F_{\mathbf{a}}(t) \leq F_{\mathbf{b}}(t)$ for all $t \in \mathbb{R}$.*

(vi) $|I| = |J|$ *and $G_{\mathbf{a}}(t) \geq G_{\mathbf{b}}(t)$ for all $t \in \mathbb{R}$.*

Moreover, if $I = J$ and $a_i \geq b_i$ for all $i \in I$, then for each $z \in \mathbb{R}$:

- $G_{(a_i)_{i \in I : a_i \geq z}}(t) \geq G_{(b_j)_{j \in J : b_j \geq z}}(t)$ *for all $t \in \mathbb{R}$.*

- $F_{(a_i)_{i \in I : a_i < z}}(t) \leq F_{(b_j)_{j \in J : b_j < z}}(t)$ *for all $t \in \mathbb{R}$.*

In particular, if $\mu_{\mathbf{a}}$ stochastically dominates $\mu_{\mathbf{b}}$, then $\mu_{\mathbf{a}}(K) = \mu_{\mathbf{b}}(K)$ for all $K \subseteq \mathbb{R}$ containing the supports of $\mu_{\mathbf{a}}$ and $\mu_{\mathbf{b}}$ (i.e., they have the same total mass). On the other hand if $\mu_{\mathbf{e}} = 0$ (that is $\mathbf{e} = (e_i)_{i \in \emptyset}$), then $F_{\mathbf{e}} = 0 \leq F_{\mathbf{d}}$ and $G_{\mathbf{d}} \geq 0 = G_{\mathbf{e}}$ for all $\mathbf{d}$, that is $\mu_{\mathbf{e}}$ lower dominates and is upper dominated by every measure $\mu_{\mathbf{d}}$. Therefore, if $\mu_{\mathbf{d}}$ stochastically dominates or is stochastically dominated by $\mu_{\mathbf{e}}$, it follows that $\mu_{\mathbf{d}} = 0$ (from $0 \leq F_{\mathbf{d}} \leq F_{\mathbf{e}} = 0$ and $0 \leq G_{\mathbf{d}} \leq G_{\mathbf{e}} = 0$, respectively). This allows to conclude that in any case stochastic dominance between $\mu_{\mathbf{a}}$ and $\mu_{\mathbf{b}}$ implies that they have the same total mass.

11.2 Chain Rules for Dini Derivatives

Let $a, b \in \mathbb{R}$ with $a < b$. If $f, g : (a, b) \to \mathbb{R}$, set

$$\limsup_{x \to a^+} f(x) \equiv \lim_{\delta \to 0^+} \sup_{h \in (0,\delta)} f(a + h) = \inf_{\delta > 0} \sup_{h \in (0,\delta)} f(a + h),$$

$$\liminf_{x \to a^+} f(x) \equiv \lim_{\delta \to 0^+} \inf_{h \in (0,\delta)} f(a + h) = \sup_{\delta > 0} \inf_{h \in (0,\delta)} f(a + h),$$

$$\limsup_{x \to b^-} f(x) \equiv \lim_{\delta \to 0^+} \sup_{h \in (0,\delta)} f(b - h) = \inf_{\delta > 0} \sup_{h \in (0,\delta)} f(b - h),$$

$$\liminf_{x \to b^-} f(x) \equiv \lim_{\delta \to 0^+} \inf_{h \in (0,\delta)} f(b - h) = \sup_{\delta > 0} \inf_{h \in (0,\delta)} f(b - h).$$

These limits always exist in $[-\infty, +\infty]$,[46] with the conventions:

$$(-\infty) \dot{+} (+\infty) = (+\infty) \dot{+} (-\infty) = +\infty = (-\infty) \dot{-} (-\infty) = (+\infty) \dot{-} (+\infty),$$

$$(-\infty) \mathbin{\underset{.}{+}} (+\infty) = (+\infty) \mathbin{\underset{.}{+}} (-\infty) = -\infty = (-\infty) \mathbin{\underset{.}{-}} (-\infty) = (+\infty) \mathbin{\underset{.}{-}} (+\infty).$$

It is easy to see that $\limsup_{x \to a^+} f(x) = \sup \{ \limsup_{n \to \infty} f(x_n) : (a, b) \ni x_n \to a^+ \}$, and that there exists $(a, b) \ni \bar{x}_n \to a^+$ such that $\lim_{n \to \infty} f(\bar{x}_n) = \limsup_{x \to a^+} f(x)$.[47]

Let $f : [a, b] \to \mathbb{R}$, and set

$$D^+ f(c) \equiv \limsup_{x \to c^+} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad D_+ f(c) \equiv \liminf_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

for all $c \in [a, b)$, and, analogously,

$$D^- f(c) \equiv \limsup_{x \to c^-} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad D_- f(c) \equiv \liminf_{x \to c^-} \frac{f(x) - f(c)}{x - c}$$

for all $c \in (a, b]$. It is easy to see that, if $c \in [a, b)$ is a local maximum, then

$$D_+ f(c) \leq D^+ f(c) \leq 0,$$

and, analogously, if $c \in (a, b]$ is a local maximum, then

$$0 \leq D_- f(c) \leq D^- f(c).$$

Next we provide a chain rule that will be useful in the sequel:

Proposition 14 *Let $v : [\alpha, \beta] \to [a, b]$ be strictly increasing, onto, and concave. Then, given any $f : [a, b] \to \mathbb{R}$, we have:*

(i) $D^+ (f \circ v)(\gamma) = v'_+(\gamma) D^+ f(v(\gamma))$ *provided either $\gamma \in (\alpha, \beta)$ or $\gamma = \alpha$ and either $v'_+(\alpha) \neq +\infty$ or $D^+(f)(v(\alpha)) > 0$.*

(ii) $D_+ (f \circ v)(\gamma) = v'_+(\gamma) D_+ f(v(\gamma))$ *provided either $\gamma \in (\alpha, \beta)$ or $\gamma = \alpha$ and either $v'_+(\alpha) \neq +\infty$ or $D_+ f(v(\alpha)) > 0$.*

(iii) $D^- (f \circ v)(\gamma) = v'_-(\gamma) D^- f(v(\gamma))$ *provided either $\gamma \in (\alpha, \beta)$ or $\gamma = \beta$ and $v'_-(\beta) \neq 0$.*

(iv) $D_- (f \circ v)(\gamma) = v'_-(\gamma) D_- f(v(\gamma))$ *provided either $\gamma \in (\alpha, \beta)$ or $\gamma = \beta$ and $v'_-(\beta) \neq 0$.*

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[46]To be precise, we should write $d > \delta > 0$, where $d \in (0, b - a)$, rather than $\delta > 0$. But, no confusion should arise.

[47]With analogous results for the other limits.

11.3 Weakly Increasing Transformations of Expected Values

Let K be a nontrivial interval in the real line, I a non-empty finite set, and $\succsim$ be a binary relation on the hypercube K^I.

Axiom 1 $\succsim$ *is complete and transitive.*

Axiom 2 *Let* $x, y \in K^I$. *If* $x_i \geq y_i$ *for all* i *in* I, *then* $x \succsim y$.

Axiom 3 *For all* $x, y, z \in K^I$, *the sets* $\{\alpha \in [0,1] : \alpha x + (1 - \alpha) z \succsim y\}$ *and* $\{\alpha \in [0,1] : \alpha x + (1 - \alpha) z \precsim y\}$ *are closed.*

Axiom 4 *Let* $x, y \in K^I$. *If* $\alpha x + (1 - \alpha) z \succ \alpha y + (1 - \alpha) z$ *for some* α *in* $(0,1]$ *and* z *in* K^I, *then* $\beta x + (1 - \beta) w \succsim \beta y + (1 - \beta) w$ *for all* β *in* $(0,1]$ *and* w *in* K^I.

Axiom 5 *Let* $x, y \in K^I$. *If* $x \succsim y$, *then* $\alpha x + (1 - \alpha) z \succsim \alpha y + (1 - \alpha) z$ *for all* α *in* $(0,1]$ *and* z *in* K^I.

Passing to the contrapositive shows that the classical independence Axiom 5 implies Axiom 4 (under completeness).

Denote by $\Pi(I)$ the set of all permutations of I.

Axiom 6 $x \sim x \circ \pi$, *for all* $x \in K^I$ *and each* $\pi \in \Pi(I)$.

Lemma 8 *A binary relation* $\succsim$ *on* K^I *satisfies Axioms 1-4 if and only if there exist a probability measure* m *on* I *and a continuous and (weakly) increasing function* $\psi : K \to \mathbb{R}$ *such that*

$$x \succsim y \Leftrightarrow \psi(m \cdot x) \geq \psi(m \cdot y). \tag{32}$$

In this case, $\succsim$ *satisfies Axiom 6 if and only if (32) holds for the uniform* m *(i.e.* $m_i \equiv 1/|I|$ *for all* $i \in I$*).*

Proof of Lemma 8. If $\succsim$ is trivial take any m and any constant ψ (in particular, the uniform m will do).

If $\succsim$ is not trivial, set

$$x \succsim^* y \Leftrightarrow \alpha x + (1 - \alpha) z \succsim \alpha y + (1 - \alpha) z \text{ for all } \alpha \in (0,1] \text{ and } z \in K^I.$$

Notice that (taking $\alpha = 1$) the above definition guarantees that $x \succsim^* y$ implies $x \succsim y$.

Next we show that $\succsim^*$ is complete. In fact, $x \not\succsim^* y$ implies $\alpha x + (1 - \alpha) z \prec \alpha y + (1 - \alpha) z$ for some $\alpha \in (0,1]$ and $z \in K^I$, but $\succsim$ satisfies Axiom 4, thus $\alpha x + (1 - \alpha) z \precsim \alpha y + (1 - \alpha) z$ for all $\alpha \in (0,1]$ and $z \in K^I$, that is $y \succsim^* x$. Moreover, $\succsim^*$ is transitive. In fact, $x \succsim^* y$ and $y \succsim^* w$ implies $\alpha x + (1 - \alpha) z \succsim \alpha y + (1 - \alpha) z$ and $\alpha y + (1 - \alpha) z \succsim \alpha w + (1 - \alpha) z$ for all $\alpha \in (0,1]$ and $z \in K^I$, then $\alpha x + (1 - \alpha) z \succsim \alpha w + (1 - \alpha) z$ for all $\alpha \in (0,1]$ and $z \in K^I$, thus $x \succsim^* w$. Then $\succsim^*$ satisfies Axiom 1.

Next we show that $\succsim^*$ satisfies Axiom 2. Let $x, y \in K^I$. If $x_i \geq y_i$ for all i in I, then $\alpha x_i + (1 - \alpha) z_i \geq \alpha y_i + (1 - \alpha) z_i$ for all $i \in I$, $\alpha \in (0,1]$, and $z \in K^I$, but $\succsim$ satisfies Axiom 2, thus $\alpha x + (1 - \alpha) z \succsim \alpha y + (1 - \alpha) z$ for all $\alpha \in (0,1]$ and $z \in K^I$, that is $x \succsim^* y$.

Next we show that $\succsim^*$ satisfies Axiom 3. Let $x, y, w \in K^I$, $\{\beta_k\}_{k \in \mathbb{N}} \subseteq [0,1]$ be such that $\beta_k x + (1 - \beta_k) y \succsim^* w$ for all $k \in \mathbb{N}$, and $\beta_k \to \beta$ as $k \to \infty$. Arbitrarily choose $\alpha \in (0,1]$ and $z \in K^I$, then

$$\alpha(\beta_k x + (1 - \beta_k) y) + (1 - \alpha) z \succsim \alpha w + (1 - \alpha) z \text{ for all } k \in \mathbb{N},$$

but $\alpha \left(\beta_k x + (1 - \beta_k) y \right) + (1 - \alpha) z = \beta_k \left(\alpha x + (1 - \alpha) z \right) + (1 - \beta_k) \left(\alpha y + (1 - \alpha) z \right)$, hence

$$\beta_k \left(\alpha x + (1 - \alpha) z \right) + (1 - \beta_k) \left(\alpha y + (1 - \alpha) z \right) \succsim \alpha w + (1 - \alpha) z \text{ for all } k \in \mathbb{N}.$$

Since $\succsim$ satisfies Axiom 3, then we can pass to the limit as $k \to \infty$ and find $\beta \left(\alpha x + (1 - \alpha) z \right) + (1 - \beta) \left(\alpha y + (1 - \alpha) z \right) \succsim \alpha w + (1 - \alpha) z$, that is

$$\alpha \left(\beta x + (1 - \beta) y \right) + (1 - \alpha) z \succsim \alpha w + (1 - \alpha) z.$$

Since this is true for all $\alpha \in (0, 1]$ and $z \in K^I$, it implies $\beta x + (1 - \beta) y \succsim^* w$. Therefore $\{ \gamma \in [0, 1] : \gamma x + (1 - \gamma) y \succsim^* w \}$ is closed. Replacing $\succsim^*$ with $\precsim^*$ (and $\succsim$ with $\precsim$), the same can be proved for the set $\{ \gamma \in [0, 1] : \gamma x + (1 - \gamma) y \precsim^* w \}$.

Next we show that $\succsim^*$ satisfies Axiom 5. Let $x \succsim^* y$, α, β in $(0, 1]$, and w, z in K^I.

$$\alpha \left(\beta x + (1 - \beta) w \right) + (1 - \alpha) z = \begin{cases} x & \text{if } \alpha\beta = 1 \text{ (i.e. } \alpha = \beta = 1), \\ \alpha\beta x + (1 - \alpha\beta) \left(\frac{\alpha(1-\beta)}{1-\alpha\beta} w + \frac{(1-\alpha)}{1-\alpha\beta} z \right) & \text{else.} \end{cases}$$

Notice that, in the second case, $\frac{\alpha(1-\beta)}{1-\alpha\beta} w + \frac{(1-\alpha)}{1-\alpha\beta} z \in K^I$ is a *bona fide* convex combination. Thus, if $\alpha\beta \neq 1$, since $x \succsim^* y$, we can conclude

$$\alpha\beta x + (1 - \alpha\beta) \left(\frac{\alpha(1-\beta)}{1-\alpha\beta} w + \frac{(1-\alpha)}{1-\alpha\beta} z \right) \succsim \alpha\beta y + (1 - \alpha\beta) \left(\frac{\alpha(1-\beta)}{1-\alpha\beta} w + \frac{(1-\alpha)}{1-\alpha\beta} z \right)$$

that is

$$\alpha \left(\beta x + (1 - \beta) w \right) + (1 - \alpha) z \succsim \alpha \left(\beta y + (1 - \beta) w \right) + (1 - \alpha) z. \tag{33}$$

Clearly, (33) descends from $x \succsim^* y$ also if $\alpha\beta = 1$. Therefore $x \succsim^* y$ implies (33) for all α, β in $(0, 1]$ and w, z in K^I; *a fortiori* it implies $\beta x + (1 - \beta) w \succsim^* \beta y + (1 - \beta) w$ for all β in $(0, 1]$ and w in K^I.

Finally, since $x \succsim^* y$ implies $x \succsim y$ and both relations are complete, non-triviality of $\succsim$ implies non-triviality of $\succsim^*$.

By the Anscombe-Aumann Theorem there exists a (unique) probability measure m on I such that $x \succsim^* y$ if and only if $m \cdot x \geq m \cdot y$; in particular,

$$m \cdot x \geq m \cdot y \Rightarrow x \succsim y. \tag{34}$$

Consider the restriction of $\succsim$ to the set of all constant elements of K^I and the usual identification of this set with K.[48] Such restriction is clearly complete, transitive, and monotonic. Next we show that it is also topologically continuous. Let $t_n, t, r \in K$ be such that $t_n \to t$ as $n \to \infty$ and $t_n \succsim r$ (resp. $t_n \precsim r$) for all $n \in \mathbb{N}$. Since t_n is converging to $t \in K$, there exist $\tau, T \in K$ ($\tau < T$) such that $t_n, t \in [\tau, T]$ for all $n \in \mathbb{N}$. Let $\alpha_n = (T - \tau)^{-1} (t_n - \tau)$ for all $n \in \mathbb{N}$. Clearly $\{\alpha_n\}_{n \in \mathbb{N}} \subseteq [0, 1]$, $\alpha_n \to (T - \tau)^{-1} (t - \tau) = \alpha$ as $n \to \infty$, $t_n = \alpha_n T + (1 - \alpha_n) \tau$ and $t = \alpha T + (1 - \alpha) \tau$. Axiom 3 and $\alpha_n T + (1 - \alpha_n) \tau = t_n \succsim r$ (resp. $t_n \precsim r$) imply $t = \alpha T + (1 - \alpha) \tau \succsim r$ (resp. $t \precsim r$).

Therefore, there exists a continuous and increasing function $\psi : K \to \mathbb{R}$ such that $\vec{t} \succsim \vec{r}$ if and only if $\psi(t) \geq \psi(r)$. Let m be any probability measure that satisfies (34), then $x \sim \overline{m \cdot x}$ for every $x \in K^I$, and $x \succsim y$ if and only if $\overline{m \cdot x} \succsim \overline{m \cdot y}$ if and only if $\psi(m \cdot x) \geq \psi(m \cdot y)$. This proves that Axioms 1-4 are sufficient for representation (32). The converse is trivial.

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[48]With the usual convention of denoting by t both the real number $t \in K$ and the constant element $\vec{t}$ of K^I taking value t for all $i \in I$.

Assume that ψ and m represent $\succsim$ in the sense of (32). Notice that the set O of all probabilities p such that ψ and p represent $\succsim$ in the sense of (32) coincides with the set of all probabilities q is such that $q \cdot x \geq q \cdot y$ implies $x \succsim y$.[49] Let $p, q \in O$ and α in $[0, 1]$, then $(\alpha p + (1 - \alpha) q) \cdot x \geq (\alpha p + (1 - \alpha) q) \cdot y$ implies $\alpha (p \cdot x) + (1 - \alpha) (q \cdot x) \geq \alpha (p \cdot y) + (1 - \alpha) (q \cdot y)$, hence either $p \cdot x \geq p \cdot y$ or $q \cdot x \geq q \cdot y$, in any case $x \succsim y$. Therefore O is convex.

Assume $\succsim$ satisfies Axiom 6, and let $m \in O$. For each $\pi \in \Pi(I)$ and each x in K^I, $x \sim x \circ \pi$ implies $\psi(m \cdot x) = \psi(m \cdot (x \circ \pi))$, but $m \cdot (x \circ \pi) = \sum_{i \in I} m_i x_{\pi(i)} = \sum_{i \in I} m_{\pi^{-1}(i)} x_{\pi(\pi^{-1}(i))} = (m \circ \pi^{-1}) \cdot x$. Therefore $\psi(m \cdot x) = \psi((m \circ \pi^{-1}) \cdot x)$ for all $x \in K^I$ and each $\pi \in \Pi(I)$. Then, for each $\sigma \in \Pi(I)$, $x \succsim y$ if and only if $\psi((m \circ \sigma) \cdot x) \geq \psi((m \circ \sigma) \cdot y)$, that is $m \circ \sigma \in O$. But O is convex, thus the uniform probability $(1/|I|) \vec{1} = \sum_{\sigma \in \Pi(I)} (1/|I|!) \, m \circ \sigma$ belongs to O. The converse is trivial. ∎

11.4 Representation Results

Proof of Lemma 2. Nontriviality, independence, and the Archimedean property of $\overset{\cdot}{\succsim}$ are immediate consequences of Axioms A.8-A.10. As to transitivity, we have $c \overset{\cdot}{\succsim} d$ and $d \overset{\cdot}{\succsim} e$ if and only if $(x_o, (x_i)_{i \in I}, d_{\{j\}}) \succsim (x_o, (x_i)_{i \in I}, c_{\{j\}})$ and $(x_o, (x_i)_{i \in I}, e_{\{j\}}) \succsim (x_o, (x_i)_{i \in I}, d_{\{j\}})$ for each $(x_o, (x_i)_{i \in I})$ in $\mathcal{X}$, and $j \notin I$. By transitivity of $\succsim$, then $(x_o, (x_i)_{i \in I}, e_{\{j\}}) \succsim (x_o, (x_i)_{i \in I}, c_{\{j\}})$, that is, $c \overset{\cdot}{\succsim} e$.

The preference $\overset{\cdot}{\succsim}$ is complete. If not $c \overset{\cdot}{\succsim} d$, then if there is a $(x_o, (x_i)_{i \in I}) \in \mathcal{X}$ and $j \notin I$ such that $(x_o, (x_i)_{i \in I}, d_{\{j\}}) \succ (x_o, (x_i)_{i \in I}, c_{\{j\}})$. By axiom A.7, there is no $(x_o, (x_i)_{i \in I}) \in \mathcal{X}$ and $j \notin I$ such that $(x_o, (x_i)_{i \in I}, c_{\{j\}}) \succ (x_o, (x_i)_{i \in I}, d_{\{j\}})$. That is, $d \overset{\cdot}{\succsim} c$. ∎

Lemma 9 *A binary relation $\succsim$ on $\mathcal{F}$ satisfies Axioms A.1-A.5 if and only if there exist a non-constant affine function $u : C \to \mathbb{R}$, a function $r : \mathcal{X} \to \mathbb{R}$, with $r(c_{I_o}) = 0$ for all $c \in C$ and $I \in \wp(N)$, and a probability P on Σ such that the functional $V : \mathcal{F} \to \mathbb{R}$ defined by*

$$V(f) = \int_S u(f_o(s)) \, dP(s) + \int_S r\left(f_o(s), (f_i(s))_{i \in I}\right) dP(s) \tag{35}$$

represents $\succsim$ and satisfies $V(\mathcal{F}) = u(C)$.

Moreover, $\left(\hat{u}, \hat{r}, \hat{P}\right)$ is another representation of $\succsim$ in the above sense if and only if $\hat{P} = P$ and there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that $\hat{u} = \alpha u + \beta$ and $\hat{r} = \alpha r$.

Proof. The von Neumann-Morgenstern Theorem guarantees that there exists an affine function $u : C \to \mathbb{R}$ such that

$$(c) \succsim (\bar{c}) \Leftrightarrow u(c) \geq u(\bar{c})$$

provided $c, \bar{c} \in C$.

Claim 9.1. For all $f \in \mathcal{F}$ there is $c^f \in C$ such that $f \sim (c^f)$.

Proof. First observe that for all $c \in C$ and all $I \in \wp(N)$, iterated application of Axiom A.5 and transitivity deliver $c_{I_o} \sim (c)$. Hence by Axiom A.3 there exist $\underline{c}, \bar{c} \in C$ such that

$$\underline{c} \precsim \left(f_o, (f_i)_{i \in I}\right) \text{ and } \left(f_o, (f_i)_{i \in I}\right) \precsim \bar{c}.$$

[49]If $p \in O$, then $p \cdot x \geq p \cdot y$ implies $\psi(p \cdot x) \geq \psi(p \cdot y)$, because ψ is increasing, and then $x \succsim y$. Conversely, observe that ψ represents $\succsim$ on K (in fact, $\vec{t} \succsim \vec{r}$ if and only if $\psi(m \cdot \vec{t}) \geq \psi(m \cdot \vec{r})$ if and only if $\psi(t) \geq \psi(r)$). If q is such that $q \cdot x \geq q \cdot y$ implies $x \succsim y$, then $x \sim \overrightarrow{q \cdot x}$ for every $x \in K^I$, and $x \succsim y$ if and only if $\overrightarrow{q \cdot x} \succsim \overrightarrow{q \cdot y}$ if and only if $\psi(q \cdot x) \geq \psi(q \cdot y)$; that is $q \in O$.

If one of the two relations is an equivalence the proof is finished. Otherwise, the above relations are strict, and, by Axiom A.2, there exist $\alpha, \beta \in (0,1)$ such that

$$(1-\alpha)\underline{c} + \alpha\bar{c} \prec \left(f_o, (f_i)_{i \in I}\right) \prec (1-\beta)\underline{c} + \beta\bar{c}.$$

and it must be $\alpha < \beta$ (u is affine on C and it represents $\succsim$ on C). By Axiom A.3 again, there exist $\lambda, \mu \in (0,1)$ such that

$$(1-\lambda)\left((1-\alpha)\underline{c} + \alpha\bar{c}\right) + \lambda\left((1-\beta)\underline{c} + \beta\bar{c}\right) \prec \left(f_o, (f_i)_{i \in I}\right) \quad \text{and}$$
$$\left(f_o, (f_i)_{i \in I}\right) \prec (1-\mu)\left((1-\alpha)\underline{c} + \alpha\bar{c}\right) + \mu\left((1-\beta)\underline{c} + \beta\bar{c}\right)$$

In particular, there exist $\alpha^* = (1-\lambda)\alpha + \lambda\beta$, $\alpha^* > \alpha$, and $\beta^* = (1-\mu)\alpha + \mu\beta$, $\beta^* < \beta$, such that, denoting $(1-\alpha)\underline{c} + \alpha\bar{c}$ by $\underline{c}\alpha\bar{c}$, we have

$$\underline{c}\alpha\bar{c} \prec \underline{c}\alpha^*\bar{c} \prec f \prec \underline{c}\beta^*\bar{c} \prec \underline{c}\beta\bar{c}$$

and $\alpha < \alpha^* < \beta^* < \beta$. (Call this argument: "shrinking".)

Set $\gamma \equiv \sup\left\{\delta \in [0,1] : \underline{c}\delta\bar{c} \prec f\right\}$. If $\delta \geq \beta^*$, then $f \prec \underline{c}\beta^*\bar{c} \precsim \underline{c}\delta\bar{c}$, and thus $\gamma \leq \beta^* < \beta$. Obviously $\gamma \geq \alpha^* > \alpha$.

Suppose $f \prec \underline{c}\gamma\bar{c}$, then $\underline{c}\alpha\bar{c} \prec f \prec \underline{c}\gamma\bar{c}$ and (shrinking) there exists $\gamma^* < \gamma$ such that $f \prec \underline{c}\gamma^*\bar{c} \prec \underline{c}\gamma\bar{c}$. Therefore $\sup\left\{\delta \in [0,1] : \underline{c}\delta\bar{c} \prec f\right\} \leq \gamma^* < \gamma$, which is absurd.

Suppose $\underline{c}\gamma\bar{c} \prec f$, then $\underline{c}\gamma\bar{c} \prec f \prec \underline{c}\beta\bar{c}$ and (shrinking) there exists $\gamma^* > \gamma$ such that $\underline{c}\gamma\bar{c} \prec \underline{c}\gamma^*\bar{c} \prec f$. Therefore $\sup\left\{\delta \in [0,1] : x\delta y \prec f\right\} \geq \gamma^* > \gamma$, which is absurd.

Conclude that $f \sim \underline{c}\gamma\bar{c} \in C$. $\qquad\qquad\square$

Claim 9.2. For all $f = \left(f_o, (f_i)_{i \in I}\right) \in \mathcal{F}$ there exists $a^f \in \mathcal{A}$ such that $f(s) \sim a^f(s)$ for all $s \in S$.

Proof. Given $f = \left(f_o, (f_i)_{i \in I}\right) \in \mathcal{F}$, denote by $\left\{A^k\right\}_{k=1}^n$ a finite partition of S in Σ that makes f_i measurable for all $i \in I_o$. For all $k = 1, ...n$, if $s, \bar{s} \in A^k$, then $\left(f_o(s), (f_i(s))_{i \in I}\right) = \left(f_o(\bar{s}), (f_i(\bar{s}))_{i \in I}\right)$; take $c^{f,k} \in C$ such that $\left(c^{f,k}\right) \sim f(s) = f(\bar{s})$. Define $a^f(s) = c^{f,k}$ if $s \in A^k$ (for $k = 1, ..., n$). The map $a : S \to C$ is a simple act, and

$$\left(a^f(s)\right) \sim f(s) \tag{36}$$

for all $s \in S$.[50] $\qquad\qquad\square$

In particular, Axiom A.2 implies $\left(a^f\right) \sim f$.

By Axiom A.1, there exist $f, g \in \mathcal{F}$ such that $f \succ g$. It follows from Claim 9.2 that $\left(a^f\right) \succ (a^g)$. Thus, the restriction of $\succsim$ to $\mathcal{A}$ (or more precisely to the subset of $\mathcal{F}$ consisting of elements of the form $f = (a)$ for some $a \in \mathcal{A} = \mathcal{A}_o$) satisfies the assumptions of the Anscombe-Aumann Theorem. Then there exist a probability P on Σ and a non-constant affine function $u : C \to \mathbb{R}$ such that

$$(a) \succsim (b) \Leftrightarrow \int_S u(a(s))\, dP(s) \geq \int_S u(b(s))\, dP(s)$$

provided $a, b \in \mathcal{A}$.[51]

For all $x \in \mathcal{X}$ set $U(x) \equiv u(c^x)$ provided $c^x \in C$ and $x \sim (c^x)$, clearly, U is well defined (on $\mathcal{X}$). Moreover, as observed, $c_{I_o} \sim (c)$ for all $c \in C$ and $I \in \wp(N)$, thus $U(c_{I_o}) = u(c)$.

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[50]In fact, $\left(a^f(s)\right) = \left(c^{f,k}\right) \sim f(s)$ provided $s \in A^k$.

[51]Notice that u represents $\succsim$ on C, hence w.l.o.g. this u is the same u we considered at the very beginning of this proof.

Let f, g in $\mathcal{F}$ and take a^f and a^g in $\mathcal{A}$ such that $\left(a^f(s)\right) \sim f(s)$ and $\left(a^g(s)\right) \sim g(s)$ for every s in S (see Claim 9.2). Then

$$\left(f_o, (f_i)_{i \in I}\right) \succsim \left(g_o, (g_j)_{j \in J}\right) \Leftrightarrow \left(a^f\right) \succsim \left(a^g\right) \Leftrightarrow$$

$$\Leftrightarrow \int_S u\left(a^f(s)\right) dP(s) \geq \int_S u\left(a^g(s)\right) dP(s) \Leftrightarrow$$

$$\Leftrightarrow \int_S U\left(f_o(s), (f_i(s))_{i \in I}\right) dP(s) \geq \int_S U\left(g_o(s), (g_j(s))_{j \in J}\right) dP(s)$$

That is, the function defined, for all $\left(f_o, (f_i)_{i \in I}\right) \in \mathcal{F}$, by

$$V\left(f_o, (f_i)_{i \in I}\right) \equiv \int_S U\left(f_o(s), (f_i(s))_{i \in I}\right) dP(s)$$

represents $\succsim$ on $\mathcal{F}$. Notice that $V\left(x_o, (x_i)_{i \in I}\right) = U\left(x_o, (x_i)_{i \in I}\right)$ for all $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$. Set $r\left(x_o, (x_i)_{i \in I}\right) \equiv U\left(x_o, (x_i)_{i \in I}\right) - u(x_o)$ for all $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$. Then $r\left(c_{I_o}\right) = U\left(c_{I_o}\right) - u(c) = 0$ for all $I \in \wp(N)$ and $c \in C$, and

$$V\left(f_o, (f_i)_{i \in I}\right) = \int_S \left[u\left(f_o(s)\right) + r\left(f_o(s), (f_i(s))_{i \in I}\right)\right] dP(s) \tag{37}$$

for all $\left(f_o, (f_i)_{i \in I}\right) \in \mathcal{F}$. Which delivers representation (35). Moreover, for all $c \in C$, $u(c) = V(c) \in V(\mathcal{F})$ and conversely, for all $f \in \mathcal{F}$, $V(f) = V\left(c^f\right) = u\left(c^f\right) \in u(C)$; i.e. $V(\mathcal{F}) = u(C)$.

Conversely, assume that there exist a non-constant affine function $u : C \to \mathbb{R}$, a function $r : \mathcal{X} \to \mathbb{R}$ with $r\left(c_{I_o}\right) = 0$ for all $c \in C$ and $I \in \wp(N)$, and a probability P on Σ, such that representation (35) holds and $V(\mathcal{F}) = u(C)$.[52] Then:

(i) $V\left(x_o, (x_i)_{i \in I}\right) = u(x_o) + r\left(x_o, (x_i)_{i \in I}\right)$ for all $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$,

(ii) $r(c) = 0$ for all $c \in C$,

(iii) $V(a) = \int_S u(a(s)) dP(s)$ for all $a \in \mathcal{A}$,

(iv) $V\left(c_{I_o}\right) = u(c)$ for all $c \in C$ and $I \in \wp(N)$.

Proving necessity of the axioms for the representation is a standard exercise. We report it just for the sake of completeness. Completeness and transitivity of $\succsim$ are obvious, non-triviality descends from (iv) above and the fact that u is not constant: Axiom A.1 holds. Let $f, g \in \mathcal{F}$ be such that $f(s) \succsim g(s)$ for all $s \in S$, then by (i)

$$u\left(f_o(s)\right) + r\left(f_o(s), (f_i(s))_{i \in I}\right) \geq u\left(g_o(s)\right) + r\left(g_o(s), (g_j(s))_{j \in J}\right)$$

for all $s \in S$, thus

$$\int_S \left[u\left(f_o(s)\right) + r(f(s))\right] dP(s) \geq \int_S \left[u\left(g_o(s)\right) + r(g(s))\right] dP(s)$$

which together with representation (35) delivers $f \succsim g$: Axiom A.2 holds. Axiom A.3 holds because of (iv), $V(\mathcal{F}) = u(C)$, and affinity of u. Axiom A.4 holds because of (iii). Finally, for all $I \in \wp(N)$, $j \in N \backslash I$, and $c \in C$, by (iv), $V\left(c_{I_o}\right) = u(c) = V\left(c_{I_o \cup \{j\}}\right)$ and Axiom A.5 holds.

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[52]Notice that $r \circ f : S \to \mathbb{R}$ is a simple and measurable function for all $f \in \mathcal{F}$, hence the integral in (35) is well defined.

Let $\hat{u}: C \to \mathbb{R}$ a non-constant affine function, $\hat{r}: \mathcal{X} \to \mathbb{R}$ a function with $\hat{r}(c_{I_o}) = 0$ for all $I \in \wp(N)$ and $c \in C$, and $\hat{P}$ be a probability on Σ, such that the functional $\hat{V}: \mathcal{F} \to \mathbb{R}$, defined by

$$\hat{V}(f) = \int_S \left[\hat{u}(f_o(s)) + \hat{r}\left(f_o(s), (f_i(s))_{i \in I}\right) \right] dP(s) \quad \forall f \in \mathcal{F}$$

represents $\succsim$ and satisfies $\hat{V}(\mathcal{F}) = \hat{u}(C)$. The above point (iii) implies that

$$\hat{V}(a) = \int_S \hat{u}(a(s)) d\hat{P}(s) \quad \forall a \in \mathcal{A}$$

is an Anscombe-Aumann representation of $\succsim$ on $\mathcal{A}$. Therefore $\hat{P} = P$, and there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that $\hat{u} = \alpha u + \beta$. For all $x \in \mathcal{X}$, take $c \in C$ such that $\hat{V}(x) = \hat{u}(c)$, then, by (iv), $x \sim (c)$ and, by (iv) again, $V(x) = u(c)$. Points (i) and (iv) imply

$$\hat{r}(x) = \hat{V}(x) - \hat{u}(x_o) = \hat{u}(c) - \hat{u}(x_o) = \alpha u(c) + \beta - \alpha u(x_o) - \beta$$
$$= \alpha(u(c) - u(x_o)) = \alpha(V(x) - u(x_o)) = \alpha r(x)$$

that is, $\hat{r} = \alpha r$. Conversely, if there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that $\hat{u} = \alpha u + \beta$, $\hat{r} = \alpha r$, and $\hat{P} = P$, then $\hat{u}: C \to \mathbb{R}$ is a non-constant affine function, $\hat{r}: \mathcal{X} \to \mathbb{R}$ is a function with $\hat{r}(c_{I_o}) = 0$ for all $I \in \wp(N)$ and $c \in C$, $\hat{P}$ is a probability on Σ, and

$$\hat{V}(f) = \int_S \left[\hat{u}(f_o(s)) + \hat{r}(f(s)) \right] d\hat{P}(s)$$
$$= \int_S \left[\alpha u(f_o(s)) + \beta + \alpha r(f(s)) \right] dP(s) = \alpha V(f) + \beta$$

obviously represents $\succsim$ on $\mathcal{F}$; finally $\hat{V}(\mathcal{F}) = \alpha V(\mathcal{F}) + \beta = \alpha u(C) + \beta = \hat{u}(C)$. $\blacksquare$

Lemma 10 *Let $\succsim$ be a binary relation on $\mathcal{F}$ that satisfy Axiom A.1. The following conditions are equivalent:*

(i) $\succsim$ *satisfies Axioms A.6 and B.1.*

(ii) *If $\left(x_o, (x_i)_{i \in I}\right), \left(x_o, (y_j)_{j \in J}\right) \in \mathcal{X}$ and there is a bijection $\pi: J \to I$ such that $y_j \succsim x_{\pi(j)}$ for all $j \in J$, then $\left(x_o, (x_i)_{i \in I}\right) \succsim \left(x_o, (y_j)_{j \in J}\right)$.*

Proof of Lemma 10. (i)$\Rightarrow$(ii). Assume $\left(x_o, (x_i)_{i \in I}\right), \left(x_o, (y_j)_{j \in J}\right)$ in $\mathcal{X}$ are such that there is a bijection $\pi: J \to I$ with $y_j \succsim x_{\pi(j)}$. Set $w_j = x_{\pi(j)}$ for all $j \in J$, by Axiom A.6, $\left(x_o, (x_i)_{i \in I}\right) \sim \left(x_o, (w_j)_{j \in J}\right)$.

If $I = \emptyset$, then $J = \emptyset$ and $\left(x_o, (x_i)_{i \in I}\right) = x_o \succsim x_o = \left(x_o, (y_j)_{j \in J}\right)$. Else, we can assume $J = \{j_1, j_2, ..., j_n\}$ and, observing that $y_j \succsim w_j$ for all $j \in J$, repeated applications of Axiom B.1 deliver that

$$\left(x_o, (x_i)_{i \in I}\right) \sim (x_o, w_{j_1}, w_{j_2}, ..., w_{j_n}) \succsim (x_o, y_{j_1}, w_{j_2}, ..., w_{j_n})$$
$$\succsim (x_o, y_{j_1}, y_{j_2}, ..., w_{j_n})$$
$$...$$
$$\succsim (x_o, y_{j_1}, y_{j_2}, ..., y_{j_n}) = \left(x_o, (y_j)_{j \in J}\right)$$

48

as wanted.

(ii)⇒(i). Assume $\left(x_o, (x_i)_{i\in I}\right), \left(x_o, (y_j)_{j\in J}\right)$ in $\mathcal{X}$ are such that there is a bijection $\pi : J \to I$ such that $y_j = x_{\pi(j)}$ for all $j \in J$. Then a fortiori, $y_j \succsim x_{\pi(j)}$ and by (ii) $\left(x_o, (x_i)_{i\in I}\right) \succsim \left(x_o, (y_j)_{j\in J}\right)$. Moreover, $\pi^{-1} : I \to J$ is such that $x_i = x_{\pi(\pi^{-1}(i))} = y_{\pi^{-1}(i)}$ for all $i \in I$, in particular $x_i \succsim y_{\pi^{-1}(i)}$ for all $i \in J$, and by (ii) $\left(x_o, (x_i)_{i\in I}\right) \precsim \left(x_o, (y_j)_{j\in J}\right)$. Therefore $\left(x_o, (x_i)_{i\in I}\right) \sim \left(x_o, (y_j)_{j\in J}\right)$ and Axiom A.6 holds.

Assume $\left(x_o, (x_i)_{i\in I}\right) \in \mathcal{X}$, $j \notin I$, and $\bar{c} \succsim c$. Consider $\left(x_o, (x_i)_{i\in I}, \bar{c}_{\{j\}}\right), \left(x_o, (x_i)_{i\in I}, c_{\{j\}}\right)$, and consider the identity $\pi : I \cup \{j\} \to I \cup \{j\}$, (ii) implies $\left(x_o, (x_i)_{i\in I}, c_{\{j\}}\right) \succsim \left(x_o, (x_i)_{i\in I}, \bar{c}_{\{j\}}\right)$, Axiom B.1 holds. ∎

Proof of Lemma 1. First observe that for all $I, J \in \wp(N)$, $(x_i)_{i\in I} \in C^I$, $(y_j)_{j\in J} \in C^J$ the following facts are equivalent:

- There is a bijection $\pi : J \to I$ such that $y_j = x_{\pi(j)}$.

- $\mu_{(x_i)_{i\in I}} = \mu_{(y_j)_{j\in J}}$.

By Lemma 9, there exist a non-constant affine function $u : C \to \mathbb{R}$, a function $r : \mathcal{X} \to \mathbb{R}$ with $r(c_{I_o}) = 0$ for all $c \in C$ and $I \in \wp(N)$, and a probability P on Σ, such that the functional $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V(f) = \int_S \left[u(f_o(s)) + r\left(f_o(s), (f_i(s))_{i\in I}\right)\right] dP(s)$$

for all $f = \left(f_o, (f_i)_{i\in I}\right) \in \mathcal{F}$, represents $\succsim$ and satisfies $V(\mathcal{F}) = u(C)$.

If $\left(x_o, (x_i)_{i\in I}\right), \left(x_o, (y_j)_{j\in J}\right) \in \mathcal{X}$ and $\mu_{(x_i)_{i\in I}} = \mu_{(y_j)_{j\in J}}$, then there is a bijection $\pi : J \to I$ such that $y_j = x_{\pi(j)}$. By Axiom A.6, $\left(x_o, (x_i)_{i\in I}\right) \sim \left(x_o, (y_j)_{j\in J}\right)$, thus $u(x_o) + r\left(x_o, (x_i)_{i\in I}\right) = u(x_o) + r\left(x_o, (y_j)_{j\in J}\right)$ and $r\left(x_o, (x_i)_{i\in I}\right) = r\left(x_o, (y_j)_{j\in J}\right)$. Therefore, for $(x_o, \mu) \in C \times \mathcal{M}(C)$ it is well posed to define

$$\varrho(x_o, \mu) = r\left(x_o, (x_i)_{i\in I}\right)$$

provided $\left(x_o, (x_i)_{i\in I}\right) \in \mathcal{X}$ and $\mu = \mu_{(x_i)_{i\in I}}$.

Finally, let $c \in C$ and $0 \le n \le |N|$. Choose $I \in \wp(N)$ with $|I| = n$, then $\varrho(c, n\delta_c) = r(c_{I_o}) = 0$. That is, ϱ is diago-null. This concludes the proof of the sufficiency part.

For the proof of necessity, set $r\left(x_o, (x_i)_{i\in I}\right) = \varrho\left(x_o, \mu_{(x_i)_{i\in I}}\right)$ for all $\left(x_o, (x_i)_{i\in I}\right) \in \mathcal{X}$ to obtain that $\succsim$ satisfies Axioms A.1-A.5 (Lemma 9). Moreover, if $\left(x_o, (x_i)_{i\in I}\right), \left(x_o, (y_j)_{j\in J}\right) \in \mathcal{X}$ and there is a bijection $\pi : J \to I$ such that $y_j = x_{\pi(j)}$ for all $j \in J$, then $\mu_{(x_i)_{i\in I}} = \mu_{(y_j)_{j\in J}}$, and hence $u(x_o) + \varrho\left(x_o, \mu_{(x_i)_{i\in I}}\right) = u(x_o) + \varrho\left(x_o, \mu_{(y_j)_{j\in J}}\right)$, that is $\left(x_o, (x_i)_{i\in I}\right) \sim \left(x_o, (y_j)_{j\in J}\right)$. Therefore Axiom A.6 holds too.

The uniqueness part immediately descends from Lemma 9 . ∎

Proof of Theorem 1. By Lemma 9, there exist a non-constant affine function $u : C \to \mathbb{R}$, a function $r : \mathcal{X} \to \mathbb{R}$ with $r(c_{I_o}) = 0$ for all $c \in C$ and $I \in \wp(N)$, and a probability P on Σ, such that the functional $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V(f) = \int_S \left[u(f_o(s)) + r\left(f_o(s), (f_i(s))_{i\in I}\right)\right] dP(s)$$

for all $f = \left(f_o, (f_i)_{i \in I}\right) \in \mathcal{F}$, represents $\succsim$ and satisfies $V(\mathcal{F}) = u(C)$.

Next we show that if $\left(x_o, (x_i)_{i \in I}\right), \left(x_o, (y_j)_{j \in J}\right) \in \mathcal{X}$ and $\mu_{(u(x_i))_{i \in I}}$ stochastically dominates $\mu_{(u(y_j))_{j \in J}}$, then $r\left(x_o, (x_i)_{i \in I}\right) \leq r\left(x_o, (y_j)_{j \in J}\right)$. Therefore, for $(x_o, \mu) \in C \times \mathcal{M}(u(C))$ it is well posed to define

$$\theta(x_o, \mu) = r\left(x_o, (x_i)_{i \in I}\right)$$

provided $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$ and $\mu = \mu_{(u(x_i))_{i \in I}}$. The obtained function θ is decreasing in the second component with respect to stochastic dominance.

If $\mu_{(u(x_i))_{i \in I}}$ stochastically dominates $\mu_{(u(y_j))_{j \in J}}$, then Lemma 7 guarantees that there exists a bijection $\pi : I \to J$ such that $u(x_i) \geq u\left(y_{\pi(i)}\right)$ for all $i \in I$, therefore $x_i \succsim y_{\pi(i)}$ for all $i \in I$. Axioms A.6 and B.1 and Lemma 10 yield $\left(x_o, (x_i)_{i \in I}\right) \precsim \left(x_o, (y_j)_{j \in J}\right)$. Then $u(x_o) + r\left(x_o, (x_i)_{i \in I}\right) \leq u(x_o) + r\left(x_o, (y_j)_{j \in J}\right)$ and $r\left(x_o, (x_i)_{i \in I}\right) \leq r\left(x_o, (y_j)_{j \in J}\right)$.

Next we show that if $(x_o, \mu), (y_o, \mu) \in C \times \mathcal{M}(u(C))$ and $u(x_o) \geq u(y_o)$, then $\theta(x_o, \mu) \geq \theta(y_o, \mu)$. Therefore, for $(z, \mu) \in \text{pim}(u(C))$ it is well posed to define

$$\varrho(z, \mu) = \theta(x_o, \mu)$$

provided $z = u(x_o)$ and ϱ is increasing in the first component and decreasing in the second component with respect to stochastic dominance.

Let $(x_o, \mu), (y_o, \mu) \in C \times \mathcal{M}(u(C))$ with $u(x_o) \geq u(y_o)$, and choose $(x_i)_{i \in I}$ such that $\mu = \mu_{(u(x_i))_{i \in I}}$. Axiom B.2 implies

$$\frac{1}{2} c\left(x_o, (x_i)_{i \in I}\right) + \frac{1}{2} y_o \succsim \frac{1}{2} x_o + \frac{1}{2} c\left(y_o, (x_i)_{i \in I}\right)$$

provided $\left(x_o, (x_i)_{i \in I}\right) \sim c\left(x_o, (x_i)_{i \in I}\right) \in C$ and $\left(y_o, (x_i)_{i \in I}\right) \sim c\left(y_o, (x_i)_{i \in I}\right) \in C$. That is,

$$u\left(\frac{1}{2} c\left(x_o, (x_i)_{i \in I}\right) + \frac{1}{2} y_o\right) \geq u\left(\frac{1}{2} x_o + \frac{1}{2} c\left(y_o, (x_i)_{i \in I}\right)\right)$$

whence

$$\frac{1}{2} u\left(c\left(x_o, (x_i)_{i \in I}\right)\right) + \frac{1}{2} u(y_o) \geq \frac{1}{2} u(x_o) + \frac{1}{2} u\left(c\left(y_o, (x_i)_{i \in I}\right)\right)$$

then $V\left(x_o, (x_i)_{i \in I}\right) = u\left(c\left(x_o, (x_i)_{i \in I}\right)\right)$ delivers

$$\frac{1}{2} u(x_o) + \frac{1}{2} \theta(x_o, \mu) + \frac{1}{2} u(y_o) \geq \frac{1}{2} u(x_o) + \frac{1}{2} u(y_o) + \frac{1}{2} \theta(y_o, \mu)$$

that is $\theta(x_o, \mu) \geq \theta(y_o, \mu)$, as wanted.

Finally, let $z \in u(C)$ and $0 \leq n \leq |N|$. Choose $c \in C$ such that $u(c) = z$ and $I \in \wp(N)$ with $|I| = n$, then

$$\varrho(z, n\delta_z) = \varrho\left(u(c), \sum_{i \in I} \delta_{u(c)}\right) = \theta\left(c, \sum_{i \in I} \delta_{u(c)}\right) = r(c_{I_o}) = 0.$$

That is ϱ is diago-null. This concludes the proof of the sufficiency part, since

$$r\left(x_o, (x_i)_{i \in I}\right) = \theta\left(x_o, \mu_{(u(x_i))_{i \in I}}\right) = \varrho\left(u(x_o), \mu_{(u(x_i))_{i \in I}}\right)$$

for all $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$.

For the proof of necessity, set $r\left(x_o, (x_i)_{i \in I}\right) = \varrho\left(u(x_o), \mu_{(u(x_i))_{i \in I}}\right)$ for all $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$ to obtain that $\succsim$ satisfies Axioms A.1-A.5 (Lemma 9).

Let $\left(x_o, (x_i)_{i \in I}\right)$ and $\left(x_o, (y_j)_{j \in J}\right)$ in $\mathcal{X}$ be such that there exists a bijection $\pi : J \to I$ such that $y_j \succsim x_{\pi(j)}$ for all $j \in J$. Then $u(y_j) \geq u\left(x_{\pi(j)}\right)$ for all $j \in J$, and by Lemma 7, $\sum_{j \in J} \delta_{u(y_j)}$ stochastically dominates $\sum_{i \in I} \delta_{u(x_i)}$ and $V\left(x_o, (x_i)_{i \in I}\right) \geq V\left(x_o, (y_j)_{j \in J}\right)$, thus $\left(x_o, (x_i)_{i \in I}\right) \succsim \left(x_o, (y_j)_{j \in J}\right)$. By Lemma 10, $\succsim$ satisfies Axiom A.6 and Axiom B.1.

Let $\left(x_o, (x_i)_{i \in I}\right), \left(y_o, (x_i)_{i \in I}\right) \in \mathcal{X}$ be such that $x_o \succsim y_o$ then $u(x_o) \geq u(y_o)$ and moreover $\varrho\left(u(x_o), \mu_{(u(x_i))_{i \in I}}\right) \geq \varrho\left(u(y_o), \mu_{(u(x_i))_{i \in I}}\right)$. Hence

$$V\left(x_o, (x_i)_{i \in I}\right) - u(x_o) \geq V\left(y_o, (x_i)_{i \in I}\right) - u(y_o)$$

$$\frac{1}{2} V\left(x_o, (x_i)_{i \in I}\right) + \frac{1}{2} u(y_o) \geq \frac{1}{2} u(x_o) + \frac{1}{2} V\left(y_o, (x_i)_{i \in I}\right)$$

$$\frac{1}{2} u\left(c\left(x_o, (x_i)_{i \in I}\right)\right) + \frac{1}{2} u(y_o) \geq \frac{1}{2} u(x_o) + \frac{1}{2} u\left(c\left(y_o, (x_i)_{i \in I}\right)\right)$$

provided $\left(x_o, (x_i)_{i \in I}\right) \sim c\left(x_o, (x_i)_{i \in I}\right) \in C$ and $\left(y_o, (x_i)_{i \in I}\right) \sim c\left(y_o, (x_i)_{i \in I}\right) \in C$. Finally, affinity of u delivers $2^{-1} c\left(x_o, (x_i)_{i \in I}\right) + 2^{-1} y_o \succsim 2^{-1} x_o + 2^{-1} c\left(y_o, (x_i)_{i \in I}\right)$. That is, Axiom B.2 holds. ∎

Proof of Proposition 1. Let $\left(\hat{u}, \hat{\varrho}, \hat{P}\right)$ be another representation of $\succsim$ in the sense of Theorem 1. Set $r\left(x_o, (x_i)_{i \in I}\right) = \varrho\left(u(x_o), \mu_{(u(x_i))_{i \in I}}\right)$ and $\hat{r}\left(x_o, (x_i)_{i \in I}\right) = \hat{\varrho}\left(\hat{u}(x_o), \mu_{(\hat{u}(x_i))_{i \in I}}\right)$ for all $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$. By Lemma 9, there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that $\hat{u} = \alpha u + \beta$, $\hat{r} = \alpha r$, and $\hat{P} = P$. Let $\left(z, \sum_{i \in I} \delta_{z_i}\right) \in \mathrm{pim}\left(\hat{u}(C)\right)$, then there exist $x = \left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$ such that $\left(z, (z_i)_{i \in I}\right) = \left(\hat{u}(x_o), (\hat{u}(x_i))_{i \in I}\right)$. Therefore $\left(z, \sum_{i \in I} \delta_{z_i}\right) = \left(\hat{u}(x_o), \mu_{(\hat{u}(x_i))_{i \in I}}\right)$, and from $\hat{r} = \alpha r$ it follows that

$$\hat{\varrho}\left(z, \sum_{i \in I} \delta_{z_i}\right) = \hat{\varrho}\left(\hat{u}(x_o), \mu_{(\hat{u}(x_i))_{i \in I}}\right) = \hat{r}\left(x_o, (x_i)_{i \in I}\right) = \alpha r\left(x_o, (x_i)_{i \in I}\right)$$

$$= \alpha \varrho\left(u(x_o), \mu_{(u(x_i))_{i \in I}}\right) = \alpha \varrho\left(\frac{z - \beta}{\alpha}, \sum_{i \in I} \delta_{\frac{z_i - \beta}{\alpha}}\right)$$

since $\hat{u} = \alpha u + \beta$ amounts to $u = \alpha^{-1}(\hat{u} - \beta)$.

Conversely, if $\hat{P} = P$, and there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that $\hat{u} = \alpha u + \beta$, and $\hat{\varrho}\left(z, \sum_{i \in I} \delta_{z_i}\right) = \alpha \varrho\left(\frac{z - \beta}{\alpha}, \sum_{i \in I} \delta_{\frac{z_i - \beta}{\alpha}}\right)$ for all $\left(z, \sum_{i \in I} \delta_{z_i}\right) \in \mathrm{pim}\left(\hat{u}(C)\right)$, then $\hat{u} : C \to \mathbb{R}$ is non-constant affine, it is easy to check that $\hat{\varrho} : \mathrm{pim}\left(\hat{u}(C)\right) \to \mathbb{R}$ is well defined, diago-null, increasing in the first component and decreasing (w.r.t. stochastic dominance) in the second, $\hat{P}$ is a probability on Σ, and

$$\hat{V}(f) = \int_S \left[\hat{u}(f_o(s)) + \hat{\varrho}\left(\hat{u}(f_o(s)), \sum_{i \in I} \delta_{\hat{u}(f_i(s))}\right)\right] dP(s)$$

$$= \int_S \left[\alpha u(f_o(s)) + \beta + \alpha \varrho\left(\frac{\hat{u}(f_o(s)) - \beta}{\alpha}, \sum_{i \in I} \delta_{\frac{\hat{u}(f_i(s)) - \beta}{\alpha}}\right)\right] dP(s)$$

$$= \alpha V(f) + \beta$$

obviously represents $\succsim$ on $\mathcal{F}$; finally $\hat{V}(\mathcal{F}) = \alpha V(\mathcal{F}) + \beta = \alpha u(C) + \beta = \hat{u}(C)$. ∎

Lemma 11 *Let $\succsim$ be a binary relation on $\mathcal{F}$ that satisfy Axioms A.1-A.5 and A.7-A.10. The following conditions are equivalent:*

(i) $\succsim$ *satisfies Axiom A.6,*

(ii) If $\left(x_o,(x_i)_{i\in I}\right),\left(x_o,(y_j)_{j\in J}\right)\in\mathcal{X}$ *and there is a bijection* $\pi:J\to I$ *such that* $y_j\overset{\cdot}{\succsim}x_{\pi(j)}$
for all $j\in J$*, then* $\left(x_o,(x_i)_{i\in I}\right)\succsim\left(x_o,(y_j)_{j\in J}\right)$*.*

Proof of Lemma 11. (i)$\Rightarrow$(ii). Let $\left(x_o,(x_i)_{i\in I}\right),\left(x_o,(y_j)_{j\in J}\right)$ in $\mathcal{X}$ be such that there is a bijection $\pi:J\to I$ with $y_j\overset{\cdot}{\succsim}x_{\pi(j)}$. Set $w_j=x_{\pi(j)}$ for all $j\in J$, by Axiom A.6 , $\left(x_o,(x_i)_{i\in I}\right)\sim\left(x_o,(w_j)_{j\in J}\right)$.

If $I=\emptyset$, then $J=\emptyset$ and $\left(x_o,(x_i)_{i\in I}\right)=x_o\succsim x_o=\left(x_o,(y_j)_{j\in J}\right)$. Else we can assume $J=\{j_1,j_2,...,j_n\}$ and, observing that $y_j\overset{\cdot}{\succsim}w_j$ for all $j\in J$, conclude that

$$\left(x_o,(x_i)_{i\in I}\right)\sim(x_o,w_{j_1},w_{j_2},...,w_{j_n})\succsim(x_o,y_{j_1},w_{j_2},...,w_{j_n})$$
$$\succsim(x_o,y_{j_1},y_{j_2},...,w_{j_n})$$
$$...$$
$$\succsim(x_o,y_{j_1},y_{j_2},...,y_{j_n})=\left(x_o,(y_j)_{j\in J}\right)$$

as wanted.

(ii)$\Rightarrow$(i). Assume $\left(x_o,(x_i)_{i\in I}\right),\left(x_o,(y_j)_{j\in J}\right)$ in $\mathcal{X}$ are such that there is a bijection $\pi:J\to I$ such that $y_j=x_{\pi(j)}$ for all $j\in J$. Then a fortiori, $y_j\overset{\cdot}{\succsim}x_{\pi(j)}$ and hence $\left(x_o,(x_i)_{i\in I}\right)\succsim\left(x_o,(y_j)_{j\in J}\right)$. Moreover, $\pi^{-1}:I\to J$ is such that $x_i=x_{\pi(\pi^{-1}(i))}=y_{\pi^{-1}(i)}$ for all $i\in I$, in particular $x_i\overset{\cdot}{\succsim}y_{\pi^{-1}(i)}$ for all $i\in J$, and hence $\left(x_o,(x_i)_{i\in I}\right)\precsim\left(x_o,(y_j)_{j\in J}\right)$. Therefore $\left(x_o,(x_i)_{i\in I}\right)\sim\left(x_o,(y_j)_{j\in J}\right)$ and Axiom A.6 holds. ∎

Lemma 11 plays for the proof of Theorem 2 the role that Lemma 10 plays for the proof of Theorem 1, as we see in the next proof.

Proof of Theorem 2. By Lemma 9, there exist a non-constant affine function $u:C\to\mathbb{R}$, a function $r:\mathcal{X}\to\mathbb{R}$ with $r(c_{I_o})=0$ for all $c\in C$ and $I\in\wp(N)$, and a probability P on Σ, such that the functional $V:\mathcal{F}\to\mathbb{R}$, defined by

$$V(f)=\int_S\left[u\left(f_o(s)\right)+r\left(f_o(s),(f_i(s))_{i\in I}\right)\right]dP(s)$$

for all $f=\left(f_o,(f_i)_{i\in I}\right)\in\mathcal{F}$, represents $\succsim$ and satisfies $V(\mathcal{F})=u(C)$. Moreover, by Axioms A.1 and A.7-A.10 there exists $v:C\to\mathbb{R}$ that represents $\overset{\cdot}{\succsim}$.

Next we show that if $\left(x_o,(x_i)_{i\in I}\right),\left(x_o,(y_j)_{j\in J}\right)\in\mathcal{X}$ and $\mu_{(v(x_i))_{i\in I}}$ stochastically dominates $\mu_{(v(y_j))_{j\in J}}$, then $r\left(x_o,(x_i)_{i\in I}\right)\leq r\left(x_o,(y_j)_{j\in J}\right)$. Therefore, for $(x_o,\mu)\in C\times\mathcal{M}(v(C))$ it is well posed to define

$$\theta(x_o,\mu)=r\left(x_o,(x_i)_{i\in I}\right)$$

provided $\left(x_o,(x_i)_{i\in I}\right)\in\mathcal{X}$ and $\mu=\mu_{(v(x_i))_{i\in I}}$. The obtained function θ is decreasing in the second component with respect to stochastic dominance.

If $\mu_{(v(x_i))_{i\in I}}$ stochastically dominates $\mu_{(v(y_j))_{j\in J}}$, then Lemma 7 guarantees that there exists a bijection $\pi:I\to J$ such that $v(x_i)\geq v\left(y_{\pi(i)}\right)$ for all $i\in I$, therefore $x_i\overset{\cdot}{\succsim}y_{\pi(i)}$ for all $i\in I$. Since $\succsim$ satisfies Axioms A.1-A.11, Lemma 11 yields $\left(x_o,(x_i)_{i\in I}\right)\precsim\left(x_o,(y_j)_{j\in J}\right)$. Then $u(x_o)+r\left(x_o,(x_i)_{i\in I}\right)\leq u(x_o)+r\left(x_o,(y_j)_{j\in J}\right)$ and $r\left(x_o,(x_i)_{i\in I}\right)\leq r\left(x_o,(y_j)_{j\in J}\right)$.

Next we show that if $(x_o, \mu), (y_o, \mu) \in C \times \mathcal{M}(v(C))$ and $v(x_o) \geq v(y_o)$, then $\theta(x_o, \mu) \geq \theta(y_o, \mu)$. Therefore, for $(z, \mu) \in \text{pim}(v(C))$ it is well posed to define

$$\varrho(z, \mu) = \theta(x_o, \mu)$$

provided $z = v(x_o)$ and ϱ is increasing in the first component and decreasing in the second component with respect to stochastic dominance.

Let $(x_o, \mu), (y_o, \mu) \in C \times \mathcal{M}(v(C))$ with $v(x_o) \geq v(y_o)$, and choose $(x_i)_{i \in I}$ such that $\mu = \mu_{(v(x_i))_{i \in I}}$. Axiom A.11 implies

$$\frac{1}{2} c\left(x_o, (x_i)_{i \in I}\right) + \frac{1}{2} y_o \succsim \frac{1}{2} x_o + \frac{1}{2} c\left(y_o, (x_i)_{i \in I}\right)$$

provided $\left(x_o, (x_i)_{i \in I}\right) \sim c\left(x_o, (x_i)_{i \in I}\right) \in C$ and $\left(y_o, (x_i)_{i \in I}\right) \sim c\left(y_o, (x_i)_{i \in I}\right) \in C$. That is,

$$u\left(\frac{1}{2} c\left(x_o, (x_i)_{i \in I}\right) + \frac{1}{2} y_o\right) \geq u\left(\frac{1}{2} x_o + \frac{1}{2} c\left(y_o, (x_i)_{i \in I}\right)\right)$$

whence

$$\frac{1}{2} u\left(c\left(x_o, (x_i)_{i \in I}\right)\right) + \frac{1}{2} u(y_o) \geq \frac{1}{2} u(x_o) + \frac{1}{2} u\left(c\left(y_o, (x_i)_{i \in I}\right)\right)$$

then $V\left(x_o, (x_i)_{i \in I}\right) = u\left(c\left(x_o, (x_i)_{i \in I}\right)\right)$ delivers

$$\frac{1}{2} u(x_o) + \frac{1}{2} \theta(x_o, \mu) + \frac{1}{2} u(y_o) \geq \frac{1}{2} u(x_o) + \frac{1}{2} u(y_o) + \frac{1}{2} \theta(y_o, \mu)$$

that is $\theta(x_o, \mu) \geq \theta(y_o, \mu)$, as wanted.

Finally, let $z \in v(C)$ and $0 \leq n \leq |N|$. Choose $c \in C$ such that $v(c) = z$ and $I \in \wp(N)$ with $|I| = n$, then

$$\varrho(z, n\delta_z) = \varrho\left(v(c), \sum_{i \in I} \delta_{v(c)}\right) = \theta\left(c, \sum_{i \in I} \delta_{v(c)}\right) = r(c_{I_o}) = 0.$$

That is ϱ is diago-null. This concludes the proof of the sufficiency part, since

$$r\left(x_o, (x_i)_{i \in I}\right) = \theta\left(x_o, \mu_{(v(x_i))_{i \in I}}\right) = \varrho\left(v(x_o), \mu_{(v(x_i))_{i \in I}}\right)$$

for all $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$.

For the proof of necessity, set $r\left(x_o, (x_i)_{i \in I}\right) = \varrho\left(v(x_o), \mu_{(v(x_i))_{i \in I}}\right)$ for all $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$ to obtain that $\succsim$ satisfies Axioms A.1-A.5 (Lemma 9). Moreover, since v is non-constant affine and it represents $\overset{\cdot}{\succsim}$, then $\succsim$ satisfies Axioms A.7-A.10.

Let $\left(x_o, (x_i)_{i \in I}\right)$ and $\left(x_o, (y_j)_{j \in J}\right)$ in $\mathcal{X}$ be such that there exists a bijection $\pi: J \to I$ such that $y_j = x_{\pi(j)}$ for all $j \in J$. Then $\sum_{j \in J} \delta_{v(y_j)} = \sum_{j \in J} \delta_{v\left(x_{\pi(j)}\right)} = \sum_{i \in I} \delta_{v(x_i)}$ and $V\left(x_o, (x_i)_{i \in I}\right) = V\left(x_o, (y_j)_{j \in J}\right)$, thus $\left(x_o, (x_i)_{i \in I}\right) \sim \left(x_o, (y_j)_{j \in J}\right)$ and Axiom A.6 holds.

Let $\left(x_o, (x_i)_{i \in I}\right), \left(y_o, (x_i)_{i \in I}\right) \in \mathcal{X}$ be such that $x_o \overset{\cdot}{\succsim} y_o$ then $v(x_o) \geq v(y_o)$ and moreover $\varrho\left(v(x_o), \mu_{(v(x_i))_{i \in I}}\right) \geq \varrho\left(v(y_o), \mu_{(v(x_i))_{i \in I}}\right)$. Hence

$$V\left(x_o, (x_i)_{i \in I}\right) - u(x_o) \geq V\left(y_o, (x_i)_{i \in I}\right) - u(y_o)$$

$$\frac{1}{2} V\left(x_o, (x_i)_{i \in I}\right) + \frac{1}{2} u(y_o) \geq \frac{1}{2} u(x_o) + \frac{1}{2} V\left(y_o, (x_i)_{i \in I}\right)$$

$$\frac{1}{2} u\left(c\left(x_o, (x_i)_{i \in I}\right)\right) + \frac{1}{2} u(y_o) \geq \frac{1}{2} u(x_o) + \frac{1}{2} u\left(c\left(y_o, (x_i)_{i \in I}\right)\right)$$

provided $\left(x_o,(x_i)_{i\in I}\right) \sim c\left(x_o,(x_i)_{i\in I}\right) \in C$ and $\left(y_o,(x_i)_{i\in I}\right) \sim c\left(y_o,(x_i)_{i\in I}\right) \in C$. Finally, affinity of u delivers $2^{-1}c\left(x_o,(x_i)_{i\in I}\right)+2^{-1}y_o \succsim 2^{-1}x_o+2^{-1}c\left(y_o,(x_i)_{i\in I}\right)$. That is, Axiom A.11 holds. ∎

Proof of Proposition 2. Let $\left(\hat{u},\hat{v},\hat{\varrho},\hat{P}\right)$ be another representation of $\succsim$ and $\overset{\cdot}{\succsim}$ in the sense of Theorem 2. Set $r\left(x_o,(x_i)_{i\in I}\right) = \varrho\left(v\left(x_o\right),\mu_{(v(x_i))_{i\in I}}\right)$ and $\hat{r}\left(x_o,(x_i)_{i\in I}\right) = \hat{\varrho}\left(\hat{v}\left(x_o\right),\mu_{(\hat{v}(x_i))_{i\in I}}\right)$ for all $\left(x_o,(x_i)_{i\in I}\right) \in \mathcal{X}$. By Lemma 9, there exist $\alpha,\beta \in \mathbb{R}$ with $\alpha > 0$ such that $\hat{u} = \alpha u + \beta$, $\hat{r} = \alpha r$, and $\hat{P} = P$. Moreover, since $\hat{v}$ represents $\overset{\cdot}{\succsim}$, there are $\dot{\alpha},\dot{\beta} \in \mathbb{R}$ with $\dot{\alpha} > 0$ such that $\hat{v} = \dot{\alpha}v + \dot{\beta}$. Let $\left(z,\sum_{i\in I}\delta_{z_i}\right) \in \text{pim}\left(\hat{v}\left(C\right)\right)$, then there exist $x = \left(x_o,(x_i)_{i\in I}\right) \in \mathcal{X}$ such that $\left(z,(z_i)_{i\in I}\right) = \left(\hat{v}\left(x_o\right),(\hat{v}\left(x_i\right))_{i\in I}\right)$. Therefore $\left(z,\sum_{i\in I}\delta_{z_i}\right) = \left(\hat{v}\left(x_o\right),\mu_{(\hat{v}(x_i))_{i\in I}}\right)$, and from $\hat{r} = \alpha r$ it follows that

$$\hat{\varrho}\left(z,\sum_{i\in I}\delta_{z_i}\right) = \hat{\varrho}\left(\hat{v}\left(x_o\right),\mu_{(\hat{v}(x_i))_{i\in I}}\right) = \hat{r}\left(x_o,(x_i)_{i\in I}\right) = \alpha r\left(x_o,(x_i)_{i\in I}\right)$$

$$= \alpha\varrho\left(v\left(x_o\right),\mu_{(v(x_i))_{i\in I}}\right) = \alpha\varrho\left(\frac{z-\dot{\beta}}{\dot{\alpha}},\sum_{i\in I}\delta_{\frac{z_i-\dot{\beta}}{\dot{\alpha}}}\right)$$

since $\hat{v} = \dot{\alpha}v + \dot{\beta}$ amounts to $v = \dot{\alpha}^{-1}\left(\hat{v} - \dot{\beta}\right)$.

Conversely, if $\hat{P} = P$, and there exist $\alpha,\beta,\dot{\alpha},\dot{\beta} \in \mathbb{R}$ with $\alpha,\dot{\alpha} > 0$ such that $\hat{u} = \alpha u + \beta$, $\hat{v} = \dot{\alpha}v + \dot{\beta}$, and $\hat{\varrho}\left(z,\sum_{i\in I}\delta_{z_i}\right) = \alpha\varrho\left(\dot{\alpha}^{-1}\left(z-\dot{\beta}\right),\sum_{i\in I}\delta_{\dot{\alpha}^{-1}(z_i-\dot{\beta})}\right)$ for all $\left(z,\sum_{i\in I}\delta_{z_i}\right) \in \text{pim}\left(\hat{v}\left(C\right)\right)$, then $\hat{u},\hat{v} : C \to \mathbb{R}$ are non-constant affine, it is easy to check that $\hat{\varrho} : \text{pim}\left(\hat{v}\left(C\right)\right) \to \mathbb{R}$ is well defined, diago-null, increasing in the first component and decreasing (w.r.t. stochastic dominance) in the second, $\hat{P}$ is a probability on Σ, $\hat{v}$ represents $\overset{\cdot}{\succsim}$, and

$$\hat{V}(f) = \int_S\left[\hat{u}\left(f_o\left(s\right)\right) + \hat{\varrho}\left(\hat{v}\left(f_o\left(s\right)\right),\sum_{i\in I}\delta_{\hat{v}(f_i(s))}\right)\right]dP(s)$$

$$= \int_S\left[\alpha u\left(f_o\left(s\right)\right) + \beta + \alpha\varrho\left(\frac{\hat{v}\left(f_o\left(s\right)\right)-\dot{\beta}}{\dot{\alpha}},\sum_{i\in I}\delta_{\frac{\hat{v}(f_i(s))-\dot{\beta}}{\dot{\alpha}}}\right)\right]dP(s)$$

$$= \alpha V(f) + \beta$$

obviously represents $\succsim$ on $\mathcal{F}$; finally $\hat{V}\left(\mathcal{F}\right) = \alpha V\left(\mathcal{F}\right) + \beta = \alpha u\left(C\right) + \beta = \hat{u}\left(C\right)$. ∎

Proof of Proposition 3. (iii)$\Rightarrow$(i) and (ii). If $\succsim$ coincides with $\overset{\cdot}{\succsim}$ on C, then A.7-A.10 are satisfied (Lemma 9 guarantees that $\succsim$, hence $\overset{\cdot}{\succsim}$, is represented on C by an affine non-constant function $u : C \to \mathbb{R}$).

If $c \succsim c'$, then $c \overset{\cdot}{\succsim} c'$, that is

$$\left(x_o,(x_i)_{i\in I},c'_{\{j\}}\right) \succsim \left(x_o,(x_i)_{i\in I},c_{\{j\}}\right)$$

for all $\left(x_o,(x_i)_{i\in I}\right) \in \mathcal{X}$ and $j \notin I$. Then Axiom B.1 is satisfied.

Moreover, $c \succ c'$ implies that $c \overset{\cdot}{\succ} c'$, thus (by definition of $\overset{\cdot}{\succsim}$) there exist $\left(x_o,(x_i)_{i\in I}\right) \in \mathcal{X}$ and $j \notin C$ such that

$$\left(x_o,(x_i)_{i\in I},c'_{\{j\}}\right) \succ \left(x_o,(x_i)_{i\in I},c_{\{j\}}\right).$$

That is, Axiom B.3 holds.

(ii)$\Rightarrow$(iii). By Axiom B.3, $c \succsim c'$ implies that $c \overset{\cdot}{\succsim} c'$, for all $c,c' \in C$. Moreover, $c \succ c'$ implies $c \overset{\cdot}{\succ} c'$, for all $c,c' \in C$; that is, $c \overset{\cdot}{\precsim} c'$ implies $c \precsim c'$.

(i)$\Rightarrow$(iii). By Axiom B.1, $c \succsim c'$ implies that $c \overset{\cdot}{\succsim} c'$, for all $c, c' \in C$. Moreover, Lemma 9 guarantees that $\succsim$ is represented by an affine non-constant function $u : C \to \mathbb{R}$, Axioms A.1 and A.7-A.10 guarantee that $\overset{\cdot}{\succsim}$ is represented by an affine non-constant function $v : C \to \mathbb{R}$, it follows that there are $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that $v = \alpha u + \beta$, that is $\succsim$ coincides with $\overset{\cdot}{\succsim}$ on C. $\blacksquare$

Proof of Theorem 3. By Theorem 2 there exist two non-constant affine functions $u, v : C \to \mathbb{R}$, a diago-null function $\varrho : \mathrm{pim}\,(v\,(C)) \to \mathbb{R}$ increasing in the first component and decreasing (w.r.t. stochastic dominance) in the second, and a probability P on Σ, such that v represents $\overset{\cdot}{\succsim}$ and the function $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V\left(f_o, (f_i)_{i \in I}\right) = \int_S \left[u\left(f_o\,(s)\right) + \varrho\left(v\left(f_o\,(s)\right), \sum_{i \in I} \delta_{v(f_i(s))}\right) \right] dP\,(s)$$

for all $\left(f_o, (f_i)_{i \in I}\right) \in \mathcal{F}$, represents $\succsim$ and satisfies $V\,(\mathcal{F}) = u\,(C)$.

Fix $z \in v\,(C)$ and $I \in \wp\,(N) \setminus \emptyset$. Consider the relation on $v\,(C)^I$ defined by $(z_i)_{i \in I} \succsim_{z,I} (w_i)_{i \in I}$ if and only if there exist $\left(x_o, (x_i)_{i \in I}\right), \left(x_o, (y_i)_{i \in I}\right) \in \mathcal{X}$ such that $v\,(x_o) = z$, $v\,(x_i) = z_i$, $v\,(y_i) = w_i$ for all $i \in I$, and $\left(x_o, (x_i)_{i \in I}\right) \precsim \left(x_o, (y_i)_{i \in I}\right)$. $\succsim_{z,I}$ is well defined, in fact, if there exist another pair $\left(x'_o, (x'_i)_{i \in I}\right), \left(x'_o, (y'_i)_{i \in I}\right) \in \mathcal{X}$ such that $v\,(x'_o) = z$, $v\,(x'_i) = z_i$, $v\,(y'_i) = w_i$, then

$$\left(x_o, (x_i)_{i \in I}\right) \precsim \left(x_o, (y_i)_{i \in I}\right) \Leftrightarrow u\,(x_o) + \varrho\left(v\,(x_o), \mu_{v(x_i)_{i \in I}}\right) \leq u\,(x_o) + \varrho\left(v\,(x_o), \mu_{v(y_i)_{i \in I}}\right)$$

$$\Leftrightarrow \varrho\left(z, \mu_{(z_i)_{i \in I}}\right) \leq \varrho\left(z, \mu_{(w_i)_{i \in I}}\right)$$

$$\Leftrightarrow u\,(x'_o) + \varrho\left(v\,(x'_o), \mu_{v(x'_i)_{i \in I}}\right) \leq u\,(x'_o) + \varrho\left(v\,(x'_o), \mu_{v(y'_i)_{i \in I}}\right)$$

$$\Leftrightarrow \left(x'_o, (x'_i)_{i \in I}\right) \precsim \left(x'_o, (y'_i)_{i \in I}\right).$$

In particular, $(z_i)_{i \in I} \succsim_{z,I} (w_i)_{i \in I}$ if and only if $\varrho\left(z, \mu_{(z_i)_{i \in I}}\right) \leq \varrho\left(z, \mu_{(w_i)_{i \in I}}\right)$, thus $\succsim_{z,I}$ is complete, transitive, monotonic, symmetric (that is it satisfies Axioms 1, 2, 6).

Fix $z = v\,(x_o) \in v\,(C)$ and $I \in \wp\,(N) \setminus \emptyset$. Let $(z_i)_{i \in I}, (\bar{z}_i)_{i \in I} \in v\,(C)^I$. If there exist $\bar{\alpha} \in (0, 1]$ and $\bar{w} \in v\,(C)^I$ such that

$$(\bar{\alpha} z_i + (1 - \bar{\alpha})\,\bar{w}_i)_{i \in I} \succ_{z,I} (\bar{\alpha} \bar{z}_i + (1 - \bar{\alpha})\,\bar{w}_i)_{i \in I}$$

take $(x_i)_{i \in I}, (\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I} \in C^I$ such that $v\,(x_i) = z_i$, $v\,(\bar{x}_i) = \bar{z}_i$, and $v\,(\bar{y}_i) = \bar{w}_i$, then

$$\left(x_o, (\bar{\alpha} x_i + (1 - \bar{\alpha})\,\bar{y}_i)_{i \in I}\right) \prec \left(x_o, (\bar{\alpha} \bar{x}_i + (1 - \bar{\alpha})\,\bar{y}_i)_{i \in I}\right)$$

by Axiom A.13, for all $(y_i)_{i \in I} \in C^I$ and $\alpha \in (0, 1]$,

$$\left(x_o, (\alpha x_i + (1 - \alpha)\,y_i)_{i \in I}\right) \precsim \left(x_o, (\alpha \bar{x}_i + (1 - \alpha)\,y_i)_{i \in I}\right)$$

that is $(v\,(\alpha x_i + (1 - \alpha)\,y_i))_{i \in I} \succsim_{z,I} (v\,(\alpha \bar{x}_i + (1 - \alpha)\,y_i))_{i \in I}$ and

$$(\alpha z_i + (1 - \alpha)\,v\,(y_i))_{i \in I} \succsim_{z,I} (\alpha \bar{z}_i + (1 - \alpha)\,v\,(y_i))_{i \in I}$$

thus $\succsim_{z,I}$ satisfies Axiom 4, since $v\,(C)^I = \left\{ (v\,(y_i))_{i \in I} : (y_i)_{i \in I} \in C^I \right\}$.

Fix $z = v(x_o) \in v(C)$ and $I \in \wp(N) \setminus \emptyset$. For all $(z_i)_{i \in I}, (\bar{z}_i)_{i \in I}, (w_i)_{i \in I} \in v(C)^I$, take $(x_i)_{i \in I}, (\bar{x}_i)_{i \in I}, (y_i)_{i \in I} \in C$ such that $v(x_j) = z_j$, $v(\bar{x}_j) = \bar{z}_j$, and $v(y_j) = w_j$ for all $j \in I$, and notice that the sets

$$\left\{ \alpha \in [0,1] : (\alpha z_i + (1-\alpha)\bar{z}_i)_{i \in I} \succsim_{z,I} (w_i)_{i \in I} \right\}$$
$$= \left\{ \alpha \in [0,1] : \left(x_o, (\alpha x_i + (1-\alpha)\bar{x}_i)_{i \in I}\right) \precsim \left(x_o, (y_i)_{i \in I}\right) \right\} \text{ and}$$
$$\left\{ \alpha \in [0,1] : (\alpha z_i + (1-\alpha)\bar{z}_i)_{i \in I} \precsim_{z,I} (w_i)_{i \in I} \right\}$$
$$= \left\{ \alpha \in [0,1] : \left(x_o, (\alpha x_i + (1-\alpha)\bar{x}_i)_{i \in I}\right) \succsim \left(x_o, (y_i)_{i \in I}\right) \right\}$$

are closed because of Axiom A.14; thus $\succsim_{z,I}$ satisfies Axiom 3. By Lemma 8, there exists a (weakly) increasing and continuous function $\psi_{z,I} : v(C) \to \mathbb{R}$ such that

$$(z_i)_{i \in I} \succsim_{z,I} (w_i)_{i \in I} \Leftrightarrow \psi_{z,I}\left(\frac{1}{|I|}\sum_{i \in I} z_i\right) \geq \psi_{z,I}\left(\frac{1}{|I|}\sum_{i \in I} w_i\right) \tag{38}$$

Next we show that if $(z, \mu), (z, \mu') \in \operatorname{pim}(v(C)) \setminus \{(z, 0)\}$ and $\mathrm{E}(\mu) = \mathrm{E}(\mu')$,[53] then $\varrho(z, \mu) = \varrho(z, \mu')$.

- If $\mu(v(C)) = \mu'(v(C)) = n$ (which must be positive), let I be an arbitrarily chosen subset of I with cardinality n. Then there exist $(z_i)_{i \in I}, (w_i)_{i \in I} \in v(C)^I$ such that $\mu = \mu_{(z_i)_{i \in I}}$ and $\mu' = \mu_{(w_i)_{i \in I}}$. $\mathrm{E}(\mu) = \mathrm{E}(\mu')$ and (38) imply that $(z_i)_{i \in I} \sim_{z,I} (w_i)_{i \in I}$ which amounts to $\varrho\left(z, \mu_{(z_i)_{i \in I}}\right) = \varrho\left(z, \mu_{(w_i)_{i \in I}}\right)$, i.e., $\varrho(z, \mu) = \varrho(z, \mu')$.

- If $\mu(v(C)) = n$ and $\mu'(v(C)) = m$, then there exist $x = \left(x_o, (x_i)_{i \in I}\right)$ and $\left(x_o, (y_j)_{j \in J}\right)$ with $|I| = n$ and $|J| = m$ such that $z = v(x_o)$, $\mu = \mu_{(v(x_i))_{i \in I}}$ and $\mu' = \mu_{(v(y_j))_{j \in J}}$. Let $c \in C$ be such that $c \sim x$, then $c_{I_o} \sim x$, that is $(c, c_I) \sim \left(x_o, (x_i)_{i \in I}\right)$ and by Axiom A.12, given any class $\{J_i\}_{i \in I}$ of disjoint subsets of N with $|J_i| = m$ for all $i \in I$,

$$\left(c, (c_{J_i})_{i \in I}\right) \sim \left(x_o, (x_{i\,J_i})_{i \in I}\right)$$

but $\left(c, (c_{J_i})_{i \in I}\right) = c_{(\cup_i J_i) \cup \{o\}} \sim c$, hence $\left(x_o, (x_i)_{i \in I}\right) \sim \left(x_o, (x_{i\,J_i})_{i \in I}\right)$ and, setting $L \equiv \cup_i J_i$ and $\left(x_o, (x_{i\,J_i})_{i \in I}\right) \equiv (x_o, (\bar{x}_l)_{l \in L})$, obviously

$$\mathrm{E}(\mu) = \frac{1}{|I|}\sum_{i \in I} v(x_i) = \frac{1}{|I|}\sum_{i \in I}\left(\frac{1}{|J_i|}\sum_{j_i \in J_i} v(\bar{x}_{j_i})\right)$$
$$= \frac{1}{|I|}\sum_{i \in I}\left(\frac{1}{|m|}\sum_{j_i \in J_i} v(\bar{x}_{j_i})\right) = \frac{1}{|\cup_i J_i|}\sum_{i \in I}\left(\sum_{j_i \in J_i} v(\bar{x}_{j_i})\right) = \mathrm{E}\left(\mu_{(v(\bar{x}_l))_{l \in L}}\right).$$

Summing up: There exists $L \in \wp(N)$ with $|L| = mn$ and $(x_o, (\bar{x}_l)_{l \in L}) \in \mathcal{X}$ such that $\left(x_o, (x_i)_{i \in I}\right) \sim (x_o, (\bar{x}_l)_{l \in L})$ and $\mathrm{E}(\mu) = \mathrm{E}\left(\mu_{(v(\bar{x}_l))_{l \in L}}\right)$. By an identical argument we can consider an n-replica $(x_o, (\bar{y}_l)_{l \in L})$ of $\left(x_o, (y_j)_{j \in J}\right)$ (where L is the set define above) and

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[53]Here $\mathrm{E}(\mu) = \mu(\mathbb{R})^{-1}\sum_{r \in \operatorname{supp}(\mu)} r\mu(r)$, that is $|I|^{-1}\sum_{i \in I} z_i$ if $\mu = \mu_{(z_i)_{i \in I}}$.

show that $\left(x_o, (y_j)_{j\in J}\right) \sim \left(x_o, (\bar{y}_l)_{l\in L}\right)$ and $\mathrm{E}\left(\mu'\right) = \mathrm{E}\left(\mu_{(v(\bar{y}_l))_{l\in L}}\right)$. Then

$$\varrho\left(z, \mu\right) = \varrho\left(z, \mu'\right) \Leftrightarrow u\left(x_o\right) + \varrho\left(v\left(x_o\right), \mu_{(v(x_i))_{i\in I}}\right) = u\left(x_o\right) + \varrho\left(v\left(x_o\right), \mu_{(v(y_j))_{j\in J}}\right)$$

$$\Leftrightarrow \left(x_o, (x_i)_{i\in I}\right) \sim \left(x_o, (y_j)_{j\in J}\right)$$

$$\Leftrightarrow \left(x_o, (\bar{x}_l)_{l\in L}\right) \sim \left(x_o, (\bar{y}_l)_{l\in L}\right)$$

$$\Leftrightarrow \left(v\left(\bar{x}_l\right)\right)_{l\in L} \sim_{v(x_o),L} \left(v\left(\bar{y}_l\right)\right)_{l\in L}$$

and the last indifference descends from $\mathrm{E}\left(\mu_{(v(\bar{x}_l))_{l\in L}}\right) = \mathrm{E}\left(\mu\right) = \mathrm{E}\left(\mu'\right) = \mathrm{E}\left(\mu_{(v(\bar{y}_l))_{l\in L}}\right)$ and (38).

Therefore

$$\varrho\left(z, \mu\right) = \varrho\left(z, \delta_{\mathrm{E}(\mu)}\right)$$

for all $(z, \mu) \in \mathrm{pim}\left(v\left(C\right)\right) \setminus \{(z, 0)\}$. With the conventions $\mathrm{E}\left(0\right) = |\emptyset|^{-1} \sum_{i\in\emptyset} z_i = \infty$ and $\delta_\infty = 0$, we also have

$$\varrho\left(z, 0\right) = \varrho\left(z, \delta_\infty\right) = \varrho\left(z, \delta_{\mathrm{E}(0)}\right).$$

The function $\eta\left(z, t\right) = \varrho\left(z, \delta_t\right)$ for all $(z, t) \in v\left(C\right) \times \left(v\left(C\right) \cup \{\infty\}\right)$ is diago-null, increasing in the first component and decreasing in the second on $v\left(C\right)$, and $\varrho\left(z, \mu\right) = \eta\left(z, \mathrm{E}\left(\mu\right)\right)$ for all $(z, \mu) \in \mathrm{pim}\left(v\left(C\right)\right)$.

It only remains to show that η is continuously decreasing in the second component on $v\left(C\right)$. Fix $z \in v\left(C\right)$, $i \in N$ and notice that for all $t, \bar{t} \in v\left(C\right) = v\left(C\right)^{\{i\}}$

$$\eta\left(z, t\right) \geq \eta\left(z, \bar{t}\right) \Leftrightarrow \varrho\left(z, \delta_t\right) \geq \varrho\left(z, \delta_{\bar{t}}\right) \Leftrightarrow t \precsim_{z,\{i\}} \bar{t} \Leftrightarrow \psi_{z,\{i\}}\left(t\right) \leq \psi_{z,\{i\}}\left(\bar{t}\right).$$

Therefore, there exists a strictly increasing function $\vartheta : -\psi_{z,\{i\}}\left(v\left(C\right)\right) \to \mathbb{R}$ such that $\eta\left(z, t\right) = \vartheta\left(-\psi_{z,\{i\}}\left(t\right)\right)$ for all $t \in v\left(C\right)$. The proof of sufficiency is concluded by renaming η into ϱ.

To prove necessity, assume there exist two non-constant affine functions $u, v : C \to \mathbb{R}$, a diago-null function $\eta : v\left(C\right) \times \left(v\left(C\right) \cup \{\infty\}\right) \to \mathbb{R}$ increasing in the first component and continuously decreasing in the second on $v\left(C\right)$, and a probability P on Σ, such that v represents $\overset{\cdot}{\succsim}$ and the function $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V\left(f_o, (f_i)_{i\in I}\right) = \int_S \left[u\left(f_o\left(s\right)\right) + \eta\left(v\left(f_o\left(s\right)\right), \frac{1}{|I|}\sum_{i\in I} v\left(f_i\left(s\right)\right)\right)\right] dP\left(s\right)$$

for all $\left(f_o, (f_i)_{i\in I}\right) \in \mathcal{F}$, represents $\succsim$ and satisfies $V\left(\mathcal{F}\right) = u\left(C\right)$. Set $\varrho\left(z, \mu\right) = \eta\left(z, \mathrm{E}\left(\mu\right)\right)$ for all $(z, \mu) \in \mathrm{pim}\left(v\left(C\right)\right)$ (with the above convention $\mathrm{E}\left(0\right) = \infty$). It is clear that ϱ is diago-null, increasing in the first component and decreasing (w.r.t. stochastic dominance) in the second, and hence, by Theorem 2, $\succsim$ on $\mathcal{F}$ satisfies Axioms A.1-A.11. It remains to show that $\succsim$ satisfies Axioms A.12, A.13, and A.14.

As observed, for all $\left(x_o, (x_i)_{i\in I}\right) \in \mathcal{X}$, all $m \in \mathbb{N}$, and each m-replica $\left(x_o, (\bar{x}_l)_{l\in L}\right)$ of $\left(x_o, (x_i)_{i\in I}\right)$, $\mathrm{E}\left(\mu_{(v(x_i))_{i\in I}}\right) = \mathrm{E}\left(\mu_{(v(\bar{x}_l))_{l\in L}}\right)$, hence $V\left(x_o, (x_i)_{i\in I}\right) = V\left(x_o, (\bar{x}_l)_{l\in L}\right)$, which implies Axiom A.12.

As to Axiom A.13, let $\left(x_o, (x_i)_{i\in I}\right), \left(x_o, (y_i)_{i\in I}\right) \in \mathcal{X}$ and assume that

$$\left(x_o, (\bar{\alpha} x_i + (1 - \bar{\alpha}) \bar{z}_i)_{i\in I}\right) \succ \left(x_o, (\bar{\alpha} y_i + (1 - \bar{\alpha}) \bar{z}_i)_{i\in I}\right)$$

for some $\bar{\alpha}$ in $(0,1]$ and $\left(x_o, (\bar{z}_i)_{i \in I}\right) \in \mathcal{X}$,[54] then

$$u\left(x_o\right) + \eta\left(v\left(x_o\right), \frac{1}{|I|}\sum_{i \in I} v\left(\bar{\alpha}x_i + (1-\bar{\alpha})\bar{z}_i\right)\right) > u\left(x_o\right) + \eta\left(v\left(x_o\right), \frac{1}{|I|}\sum_{i \in I} v\left(\bar{\alpha}y_i + (1-\bar{\alpha})\bar{z}_i\right)\right)$$

$$\eta\left(v\left(x_o\right), \bar{\alpha}\frac{1}{|I|}\sum_{i \in I} v\left(x_i\right) + (1-\bar{\alpha})\frac{1}{|I|}\sum_{i \in I} v\left(\bar{z}_i\right)\right) > \eta\left(v\left(x_o\right), \bar{\alpha}\frac{1}{|I|}\sum_{i \in I} v\left(y_i\right) + (1-\bar{\alpha})\frac{1}{|I|}\sum_{i \in I} v\left(\bar{z}_i\right)\right)$$

hence (η is decreasing in the second component on $v\left(C\right)$)

$$\bar{\alpha}\frac{1}{|I|}\sum_{i \in I} v\left(y_i\right) + (1-\bar{\alpha})\frac{1}{|I|}\sum_{i \in I} v\left(\bar{z}_i\right) \geq \bar{\alpha}\frac{1}{|I|}\sum_{i \in I} v\left(x_i\right) + (1-\bar{\alpha})\frac{1}{|I|}\sum_{i \in I} v\left(\bar{z}_i\right)$$

$$\frac{1}{|I|}\sum_{i \in I} v\left(y_i\right) \geq \frac{1}{|I|}\sum_{i \in I} v\left(x_i\right)$$

therefore for all α in $(0,1]$ and $\left(x_o, (z_i)_{i \in I}\right) \in \mathcal{X}$

$$\alpha\frac{1}{|I|}\sum_{i \in I} v\left(y_i\right) + (1-\alpha)\frac{1}{|I|}\sum_{i \in I} v\left(z_i\right) \geq \alpha\frac{1}{|I|}\sum_{i \in I} v\left(x_i\right) + (1-\alpha)\frac{1}{|I|}\sum_{i \in I} v\left(z_i\right)$$

$$\frac{1}{|I|}\sum_{i \in I} v\left(\alpha y_i + (1-\alpha)z_i\right) \geq \frac{1}{|I|}\sum_{i \in I} v\left(\alpha x_i + (1-\alpha)z_i\right)$$

$$\eta\left(v\left(x_o\right), \frac{1}{|I|}\sum_{i \in I} v\left(\alpha y_i + (1-\alpha)z_i\right)\right) \leq \eta\left(v\left(x_o\right), \frac{1}{|I|}\sum_{i \in I} v\left(\alpha x_i + (1-\alpha)z_i\right)\right)$$

$$u\left(x_o\right) + \eta\left(v\left(x_o\right), \frac{1}{|I|}\sum_{i \in I} v\left(\alpha x_i + (1-\alpha)z_i\right)\right) \geq u\left(x_o\right) + \eta\left(v\left(x_o\right), \frac{1}{|I|}\sum_{i \in I} v\left(\alpha y_i + (1-\alpha)z_i\right)\right)$$

$$\left(x_o, (\alpha x_i + (1-\alpha)z_i)_{i \in I}\right) \succsim \left(x_o, (\alpha y_i + (1-\alpha)z_i)_{i \in I}\right)$$

as wanted.

Finally let $\left(x_o, (x_i)_{i \in I}\right), \left(x_o, (y_i)_{i \in I}\right), \left(x_o, (z_i)_{i \in I}\right) \in \mathcal{X}$ and assume $\{\alpha_n\}_{n \in \mathbb{N}} \subseteq [0,1]$, $\alpha_n \to \alpha$, and $\left(x_o, (\alpha_n x_i + (1-\alpha_n)z_i)_{i \in I}\right) \succsim \left(x_o, (y_i)_{i \in I}\right)$ for all $n \in \mathbb{N}$. Clearly, if I is empty, $\left(x_o, (\alpha x_i + (1-\alpha)z_i)_{i \in I}\right) = (x_o) = \left(x_o, (y_i)_{i \in I}\right)$, hence $\left(x_o, (\alpha x_i + (1-\alpha)z_i)_{i \in I}\right) \succsim \left(x_o, (y_i)_{i \in I}\right)$. Else, let $\psi_{v(x_o)} : v\left(C\right) \to \mathbb{R}$ be a weakly decreasing and continuous function such that for all $t, \bar{t} \in v\left(C\right)$,

$$\eta\left(v\left(x_o\right), t\right) \geq \eta\left(v\left(x_o\right), \bar{t}\right) \Leftrightarrow \psi_{v(x_o)}\left(t\right) \geq \psi_{v(x_o)}\left(\bar{t}\right)$$

(which exists since $\eta\left(v\left(x_o\right), \cdot\right)$ is continuously decreasing on $v\left(C\right)$). Then, for all $n \in \mathbb{N}$,

$$\left(x_o, (\alpha_n x_i + (1-\alpha_n)z_i)_{i \in I}\right) \succsim \left(x_o, (y_i)_{i \in I}\right)$$

$$\eta\left(v\left(x_o\right), \frac{1}{|I|}\sum_{i \in I} v\left(y_i\right)\right) \leq \eta\left(v\left(x_o\right), \frac{1}{|I|}\sum_{i \in I} v\left(\alpha_n x_i + (1-\alpha_n)z_i\right)\right)$$

$$\psi_{v(x_o)}\left(\frac{1}{|I|}\sum_{i \in I} v\left(y_i\right)\right) \leq \psi_{v(x_o)}\left(\frac{1}{|I|}\sum_{i \in I} v\left(\alpha_n x_i + (1-\alpha_n)z_i\right)\right)$$

$$\psi_{v(x_o)}\left(\frac{1}{|I|}\sum_{i \in I} v\left(y_i\right)\right) \leq \psi_{v(x_o)}\left(\frac{1}{|I|}\sum_{i \in I} \alpha_n v\left(x_i\right) + (1-\alpha_n)v\left(z_i\right)\right)$$

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[54]This cannot be the case if I is empty.

continuity of $\psi_{v(x_o)}$ delivers

$$\psi_{v(x_o)}\left(\frac{1}{|I|}\sum_{i\in I}v\left(y_i\right)\right)\leq\psi_{v(x_o)}\left(\frac{1}{|I|}\sum_{i\in I}\alpha v\left(x_i\right)+\left(1-\alpha\right)v\left(z_i\right)\right)$$

and $\left(x_o,\left(\alpha x_i+\left(1-\alpha\right)z_i\right)_{i\in I}\right)\succsim\left(x_o,\left(y_i\right)_{i\in I}\right)$.

Then the set $\left\{\alpha\in\left[0,1\right]:\left(x_o,\left(\alpha x_i+\left(1-\alpha\right)z_i\right)_{i\in I}\right)\succsim\left(x_o,\left(y_i\right)_{i\in I}\right)\right\}$ is closed, and analogous considerations hold for $\left\{\alpha\in\left[0,1\right]:\left(x_o,\left(\alpha x_i+\left(1-\alpha\right)z_i\right)_{i\in I}\right)\precsim\left(x_o,\left(y_i\right)_{i\in I}\right)\right\}$. ∎

Proof of Proposition 4. Assume that $\left(\hat{u},\hat{v},\hat{\varrho},\hat{P}\right)$ represent $\succsim$ and $\dot{\succsim}$ in the sense of Theorem 3. Set $r\left(x_o,\left(x_i\right)_{i\in I}\right)=\varrho\left(v\left(x_o\right),|I|^{-1}\sum_{i\in I}v\left(x_i\right)\right)$ and $\hat{r}\left(x_o,\left(x_i\right)_{i\in I}\right)=\hat{\varrho}\left(\hat{v}\left(x_o\right),|I|^{-1}\sum_{i\in I}\hat{v}\left(x_i\right)\right)$ for all $\left(x_o,\left(x_i\right)_{i\in I}\right)\in\mathcal{X}$, by Lemma 9 it follows that $\hat{P}=P$, and there exist $\alpha,\beta\in\mathbb{R}$ with $\alpha>0$ such that $\hat{u}=\alpha u+\beta$, and $\hat{r}=\alpha r$; moreover since $\hat{v}$ represents $\dot{\succsim}$, there are $\dot{\alpha},\dot{\beta}\in\mathbb{R}$ with $\dot{\alpha}>0$ such that $\hat{v}=\dot{\alpha}v+\dot{\beta}$. Let $\left(z,t\right)\in\hat{v}\left(C\right)^2$, $i\in N$, and $x_o,x_i\in C$ such that $z=\hat{v}\left(x_o\right)$, $t=\hat{v}\left(x_i\right)$, then $v\left(x_o\right)=\left(z-\dot{\beta}\right)/\dot{\alpha}$, $v\left(x_i\right)=\left(t-\dot{\beta}\right)/\dot{\alpha}$. Hence

$$\hat{\varrho}\left(z,t\right)=\hat{r}\left(x_o,x_i\right)=\alpha r\left(x_o,x_i\right)=\alpha\varrho\left(v\left(x_o\right),v\left(x_i\right)\right)=\alpha\varrho\left(\frac{z-\dot{\beta}}{\dot{\alpha}},\frac{t-\dot{\beta}}{\dot{\alpha}}\right)$$

and clearly $\hat{\varrho}\left(z,\infty\right)=0=\alpha\varrho\left(\frac{z-\dot{\beta}}{\dot{\alpha}},\frac{\infty-\dot{\beta}}{\dot{\alpha}}\right)$ for all $z\in\hat{v}\left(C\right)$.

The converse is trivial. In fact, if $\hat{P}=P$, and there exist $\alpha,\beta,\dot{\alpha},\dot{\beta}\in\mathbb{R}$ with $\alpha,\dot{\alpha}>0$ such that $\hat{u}=\alpha u+\beta$, $\hat{v}=\dot{\alpha}v+\dot{\beta}$, and $\hat{\varrho}\left(z,r\right)=\alpha\varrho\left(\frac{z-\dot{\beta}}{\dot{\alpha}},\frac{r-\dot{\beta}}{\dot{\alpha}}\right)$ for all $\left(z,r\right)\in\hat{v}\left(C\right)\times\left(\hat{v}\left(C\right)\cup\left\{\infty\right\}\right)$, then $\hat{u},\hat{v}:C\to\mathbb{R}$ are non-constant affine, it is easy to check that $\hat{\varrho}:\hat{v}\left(C\right)\times\left(\hat{v}\left(C\right)\cup\left\{\infty\right\}\right)\to\mathbb{R}$ is well defined, diago-null, increasing in the first component and continuously decreasing in the second on $\hat{v}\left(C\right)$, $\hat{P}$ is a probability on Σ, $\hat{v}$ represents $\dot{\succsim}$, and

$$\hat{V}\left(f\right)=\int_S\left[\hat{u}\left(f_o\left(s\right)\right)+\hat{\varrho}\left(\hat{v}\left(f_o\left(s\right)\right),|I|^{-1}\sum_{i\in I}\hat{v}\left(f_i\left(s\right)\right)\right)\right]d\hat{P}\left(s\right)$$

$$=\int_S\left[\alpha u\left(f_o\left(s\right)\right)+\beta+\alpha\varrho\left(\frac{\hat{v}\left(f_o\left(s\right)\right)-\dot{\beta}}{\dot{\alpha}},\frac{|I|^{-1}\sum_{i\in I}\hat{v}\left(f_i\left(s\right)\right)-\dot{\beta}}{\dot{\alpha}}\right)\right]dP\left(s\right)$$

$$=\alpha\int_S\left[u\left(f\left(s\right)\right)+\varrho\left(v\left(f\left(s\right)\right),|I|^{-1}\sum_{i\in I}v\left(f_i\left(s\right)\right)\right)\right]dP\left(s\right)+\beta$$

$$=\alpha V\left(f\right)+\beta$$

obviously represents $\succsim$ on $\mathcal{F}$; finally $\hat{V}\left(\mathcal{F}\right)=\alpha V\left(\mathcal{F}\right)+\beta=\alpha u\left(C\right)+\beta=\hat{u}\left(C\right)$. ∎

Proof of Theorem 4. By Theorem 2 there exist two non-constant affine functions $u,v:C\to\mathbb{R}$, a diago-null function $\varrho:\text{pim}\left(v\left(C\right)\right)\to\mathbb{R}$ increasing in the first component and decreasing (w.r.t. stochastic dominance) in the second, and a probability P on Σ, such that v represents $\dot{\succsim}$ and the function $V:\mathcal{F}\to\mathbb{R}$, defined by

$$V\left(f_o,\left(f_i\right)_{i\in I}\right)=\int_S\left[u\left(f_o\left(s\right)\right)+\varrho\left(v\left(f_o\left(s\right)\right),\sum_{i\in I}\delta_{v\left(f_i\left(s\right)\right)}\right)\right]dP\left(s\right)$$

for all $\left(f_o,\left(f_i\right)_{i\in I}\right)\in\mathcal{F}$, represents $\succsim$ and satisfies $V\left(\mathcal{F}\right)=u\left(C\right)$.

Claim 1. Let (z, μ) and (z, μ') in $\text{pim}\,(v\,(C))$ be such that $\min\,(\text{supp}\mu) \leq \min\,(\text{supp}\mu')$, $\max\,(\text{supp}\mu) \leq \max\,(\text{supp}\mu')$, then $\varrho\,(z, \mu) \geq \varrho\,(z, \mu')$.

In fact:

- If $\text{supp}\mu = \emptyset$, then $+\infty = \min\,(\text{supp}\mu) \leq \min\,(\text{supp}\mu')$, hence $\text{supp}\mu' = \emptyset$, i.e. $\mu = \mu' = 0$ and $\varrho\,(z, \mu) = \varrho\,(z, \mu') = 0$.

- If $\text{supp}\mu' = \emptyset$, then $-\infty = \max\,(\text{supp}\mu') \geq \max\,(\text{supp}\mu)$, hence $\text{supp}\mu = \emptyset$, i.e. $\mu = \mu' = 0$ and $\varrho\,(z, \mu) = \varrho\,(z, \mu') = 0$.

- Else $\text{supp}\mu, \text{supp}\mu' \neq \emptyset$. There exist $I, J \in \wp\,(N) \setminus \emptyset$ and $\left(x_o, (x_i)_{i \in I}\right), \left(x_o, (y_j)_{j \in J}\right) \in \mathcal{X}$ such that $(z, \mu) = \left(v\,(x_o), \sum_{i \in I} \delta_{v(x_i)}\right)$ and $(z, \mu') = \left(v\,(x_o), \sum_{j \in J} \delta_{v(y_j)}\right)$. Choose $i_{\min} \in \arg\min_{i \in I} v\,(x_i)$ and define $i_{\max}, j_{\min}, j_{\max}$ analogously, so that

$$\min\,(\text{supp}\mu) = v\,(x_{i_{\min}}),$$
$$\max\,(\text{supp}\mu) = v\,(x_{i_{\max}}),$$
$$\min\,(\text{supp}\mu') = v\,(y_{j_{\min}}),$$
$$\max\,(\text{supp}\mu') = v\,(y_{j_{\max}}).$$

Then for all $i \in I$ there is $j = j_{\max} \in J$ such that $y_j \dot{\succsim} x_i$, and for all $j \in J$ there is $i = i_{\min} \in I$ such that $x_i \dot{\precsim} y_j$. By Axiom C.1, $\left(x_o, (x_i)_{i \in I}\right) \succsim \left(x_o, (y_j)_{j \in J}\right)$, then

$$u\,(x_o) + \varrho\left(v\,(x_o), \sum_{i \in I} \delta_{v(x_i)}\right) \geq u\,(x_o) + \varrho\left(v\,(x_o), \sum_{j \in J} \delta_{v(y_j)}\right)$$

and $\varrho\,(z, \mu) \geq \varrho\,(z, \mu')$, as wanted.

For all $(z, t, T) \in v\,(C)^{1,2}$, set

$$\eta\,(z, t, T) = \varrho\,(z, \mu)$$

if $\mu \in \mathcal{M}\,(v\,(C))$ is such that $\min\,(\text{supp}\mu) = t$, $\max\,(\text{supp}\mu) = T$. Claim 1 implies that η is well defined and decreasing in the second and third components on $v\,(C)$.

The function η is diago-null and increasing in the first component since ϱ is. The proof of sufficiency is concluded by observing that $\varrho\,(z, \mu) = \eta\,(z, \min\,(\text{supp}\mu), \max\,(\text{supp}\mu))$ for all $(z, \mu) \in \text{pim}\,(v\,(C))$, and so

$$V\left(f_o, (f_i)_{i \in I}\right) = \int_S \left[u\,(f_o\,(s)) + \varrho\left(v\,(f_o\,(s)), \sum_{i \in I} \delta_{v(f_i(s))}\right)\right] dP\,(s)$$

$$= \int_S \left[u\,(f_o\,(s)) + \eta\left(v\,(f_o\,(s)), \min\left(\text{supp} \sum_{i \in I} \delta_{v(f_i(s))}\right), \max\left(\text{supp} \sum_{i \in I} \delta_{v(f_i(s))}\right)\right)\right] dP\,(s)$$

$$= \int_S \left[u\,(f_o\,(s)) + \eta\left(v\,(f_o\,(s)), \min_{i \in I} v\,(f_i\,(s))_{i \in I}, \max_{i \in I} v\,(f_i\,(s))\right)\right] dP\,(s)$$

for all $\left(f_o, (f_i)_{i \in I}\right) \in \mathcal{F}$.

Conversely, assume that there exist two non-constant affine functions $u, v : C \to \mathbb{R}$, a diago-null function $\eta : v\,(C)^{1,2} \to \mathbb{R}$, increasing in the first component and decreasing in the second and third on $v\,(C)$, and a probability P on Σ such that the functional $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V\left(f_o, (f_i)_{i \in I}\right) = \int_S \left[u\,(f_o\,(s)) + \eta\left(v\,(f_o\,(s)), \min_{i \in I} v\,(f_i\,(s)), \max_{i \in I} v\,(f_i\,(s))\right)\right] dP\,(s)$$

represents $\succsim$ and satisfies $V(\mathcal{F}) = u(C)$.

Set $\varrho(z,\mu) = \eta(z, \min(\mathrm{supp}\mu), \max(\mathrm{supp}\mu))$ for all $(z,\mu) \in \mathrm{pim}(u(C))$. The function ϱ is diago-null and increasing in the first component since η is. Moreover, if μ' stochastically dominates μ and $\mu' \neq 0$, then $\mu \neq 0$ and $\min(\mathrm{supp}\mu) \leq \min(\mathrm{supp}\mu')$ and $\max(\mathrm{supp}\mu) \leq \max(\mathrm{supp}\mu')$, thus

$$\eta(z, \min(\mathrm{supp}\mu), \max(\mathrm{supp}\mu)) \geq \eta\left(z, \min(\mathrm{supp}\mu'), \max(\mathrm{supp}\mu')\right) \quad \forall z \in v(C)$$

(η is decreasing in the second and third components on $v(C)$); if μ' stochastically dominates μ and $\mu' = 0$, then $\mu = 0$ and $\min(\mathrm{supp}\mu) = \min(\mathrm{supp}\mu') = +\infty$, $\max(\mathrm{supp}\mu) = \max(\mathrm{supp}\mu') = -\infty$, thus

$$\eta(z, \min(\mathrm{supp}\mu), \max(\mathrm{supp}\mu)) = \eta\left(z, \min(\mathrm{supp}\mu'), \max(\mathrm{supp}\mu')\right) \quad \forall z \in v(C).$$

In any case if μ' stochastically dominates μ, then $\varrho(z,\mu) \geq \varrho(z,\mu')$, i.e., ϱ is decreasing (w.r.t. stochastic dominance) in the second component. By Theorem 2, $\succsim$ on $\mathcal{F}$ satisfies Axioms A.1-A.11. It remains to show that $\succsim$ satisfies Axiom C.1. Let $\left(x_o, (x_i)_{i \in I}\right)$ and $\left(x_o, (y_j)_{j \in J}\right)$ in $\mathcal{X}$ be such that

(i) for all $i \in I$ there is $j \in J$ such that $y_j \dot{\succsim} x_i$,

(ii) for all $j \in J$ there is $i \in I$ such that $y_j \dot{\succsim} x_i$.

If $I = \emptyset$, by (ii), $J = \emptyset$, and (analogously, if $J = \emptyset$, by (i), $I = \emptyset$). In this case, $\left(x_o, (x_i)_{i \in I}\right) = (x_o) = \left(x_o, (y_j)_{j \in J}\right)$. Else if I and J are non-empty, then (i) implies $\max_{j \in J} v(y_j) \geq \max_{i \in I} v(x_i)$, while (ii) implies $\min_{j \in J} v(y_j) \geq \min_{i \in I} v(x_i)$, hence

$$u(x_o) + \eta\left(v(x_o), \min_{i \in I} v(x_i), \max_{i \in I} v(x_i)\right) \geq u(x_o) + \eta\left(v(x_o), \min_{j \in J} v(y_j), \max_{j \in J} v(y_j)\right)$$

thus $\left(x_o, (x_i)_{i \in I}\right) \succsim \left(x_o, (y_j)_{j \in J}\right)$. As wanted. $\blacksquare$

Proof of Proposition 5. Assume that $\left(\hat{u}, \hat{v}, \hat{\varrho}, \hat{P}\right)$ represent $\succsim$ and $\dot{\succsim}$ in the sense of Theorem 4. Set

$$r\left(x_o, (x_i)_{i \in I}\right) = \varrho\left(v(x_o), \min_{i \in I} v(x_i), \max_{i \in I} v(x_i)\right) \quad \text{and}$$

$$\hat{r}\left(x_o, (x_i)_{i \in I}\right) = \varrho\left(\hat{v}(x_o), \min_{i \in I} \hat{v}(x_i), \max_{i \in I} \hat{v}(x_i)\right)$$

for all $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$, by Lemma 9 it follows that $\hat{P} = P$, and there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that $\hat{u} = \alpha u + \beta$, and $\hat{r} = \alpha r$; moreover since $\hat{v}$ represents $\dot{\succsim}$, there are $\dot{\alpha}, \dot{\beta} \in \mathbb{R}$ with $\dot{\alpha} > 0$ such that $\hat{v} = \dot{\alpha} v + \dot{\beta}$. Let $(z, t, T) \in \hat{v}(C)^{1,2}$ with $t, T \in \hat{v}(C)$, $i, j \in N$, and $x_o, x_i, x_j \in C$ such that $z = \hat{v}(x_o)$, $t = \hat{v}(x_i)$, $T = \hat{v}(x_j)$, then $v(x_o) = \left(z - \dot{\beta}\right)/\dot{\alpha}$, $v(x_i) = \left(t - \dot{\beta}\right)/\dot{\alpha}$, $v(x_j) = \left(T - \dot{\beta}\right)/\dot{\alpha}$. Hence

$$\hat{\varrho}(z, t, T) = \hat{r}(x_o, x_i, x_j) = \alpha r(x_o, x_i, x_j) = \alpha \varrho(v(x_o), v(x_i), v(x_j)) = \alpha \varrho\left(\frac{z - \dot{\beta}}{\dot{\alpha}}, \frac{t - \dot{\beta}}{\dot{\alpha}}, \frac{T - \dot{\beta}}{\dot{\alpha}}\right)$$

and clearly $\hat{\varrho}(z, +\infty, -\infty) = 0 = \alpha\varrho\left(\dfrac{z - \dot{\beta}}{\dot{\alpha}}, \dfrac{+\infty - \dot{\beta}}{\dot{\alpha}}, \dfrac{-\infty - \dot{\beta}}{\dot{\alpha}}\right)$ for all $z \in \hat{v}(C)$.

The converse is trivial. In fact, if $\hat{P} = P$, and there exist $\alpha, \beta, \dot{\alpha}, \dot{\beta} \in \mathbb{R}$ with $\alpha, \dot{\alpha} > 0$ such that $\hat{u} = \alpha u + \beta$, $\hat{v} = \dot{\alpha}v + \dot{\beta}$, and $\hat{\varrho}(z, r, R) = \alpha\varrho\left(\dfrac{z - \dot{\beta}}{\dot{\alpha}}, \dfrac{r - \dot{\beta}}{\dot{\alpha}}, \dfrac{R - \dot{\beta}}{\dot{\alpha}}\right)$ for all $(z, r, R) \in \hat{v}(C)^{1,2}$, then $\hat{u}, \hat{v} : C \to \mathbb{R}$ are non-constant affine, it is easy to check that $\hat{\varrho} : \hat{v}(C)^{1,2} \to \mathbb{R}$ is well defined, diago-null, increasing in the first component and decreasing in the second and third on $\hat{v}(C)$, $\hat{P}$ is a probability on Σ, $\hat{v}$ represents $\overset{\cdot}{\succsim}$, and

$$\hat{V}(f) = \int_S \left[\hat{u}(f_o(s)) + \hat{\varrho}\left(\hat{v}(f_o(s)), \min_{i \in I} \hat{v}(f_i(s)), \max_{i \in I} \hat{v}(f_i(s))\right)\right] d\hat{P}(s)$$

$$= \int_S \left[\alpha u(f_o(s)) + \beta + \alpha\varrho\left(\dfrac{\hat{v}(f_o(s)) - \dot{\beta}}{\dot{\alpha}}, \dfrac{\min_{i \in I}\hat{v}(f_i(s)) - \dot{\beta}}{\dot{\alpha}}, \dfrac{\max_{i \in I}\hat{v}(f_i(s)) - \dot{\beta}}{\dot{\alpha}}\right)\right] dP(s)$$

$$= \alpha \int_S \left[u(f(s)) + \varrho\left(v(f(s)), \min_{i \in I} v(f_i(s)), \max_{i \in I} v(f_i(s))\right)\right] dP(s) + \beta$$

$$= \alpha V(f) + \beta$$

obviously represents $\succsim$ on $\mathcal{F}$; finally $\hat{V}(\mathcal{F}) = \alpha V(\mathcal{F}) + \beta = \alpha u(C) + \beta = \hat{u}(C)$. ∎

Proof of Theorem 5. By Theorem 2 there exist two non-constant affine functions $u, v : C \to \mathbb{R}$, a diago-null function $\varrho : \mathrm{pim}(v(C)) \to \mathbb{R}$ increasing in the first component and decreasing (w.r.t. stochastic dominance) in the second, and a probability P on Σ, such that v represents $\overset{\cdot}{\succsim}$ and the function $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V\left(f_o, (f_i)_{i \in I}\right) = \int_S \left[u(f_o(s)) + \varrho\left(v(f_o(s)), \sum_{i \in I} \delta_{v(f_i(s))}\right)\right] dP(s)$$

for all $\left(f_o, (f_i)_{i \in I}\right) \in \mathcal{F}$, represents $\succsim$ and satisfies $V(\mathcal{F}) = u(C)$.

For all $(z, \mu, \mu') \in \mathrm{pid}(v(C))$ set $\eta(z, \mu, \mu') = \varrho(z, \mu + \mu')$, clearly η is well defined and $\eta(z, 0, n\delta_z) = \varrho(z, n\delta_z) = 0$ for all $z \in v(C)$ and $0 \leq n \leq |N|$, that is η is diago-null. Moreover, η is increasing in its first component since ϱ is. Next we show that η is decreasing in the second component w.r.t. the lower dominance.

Let $\left(z, \sum_{i \in I} \delta_{a_i}, \sum_{l \in L} \delta_{z_l}\right), \left(z, \sum_{j \in J} \delta_{b_j}, \sum_{l \in L} \delta_{z_l}\right) \in \mathrm{pid}(u(C))$ and assume $\sum_{i \in I} \delta_{a_i}$ lower dominates $\sum_{j \in J} \delta_{b_j}$.[55]

If $I = \emptyset$ and $J = \emptyset$, then

$$\eta\left(z, \sum_{i \in I} \delta_{a_i}, \sum_{l \in L} \delta_{z_l}\right) = \eta\left(z, 0, \sum_{l \in L} \delta_{z_l}\right) = \eta\left(z, \sum_{j \in J} \delta_{b_j}, \sum_{l \in L} \delta_{z_l}\right)$$

If $I = \emptyset$ and $J \neq \emptyset$, then w.l.o.g. we can assume $J = \{j_1, ..., j_m\}$ with $b_{j_1} \leq ... \leq b_{j_m} < z$. Let $\left(x_o, (y_{j_k})_{k=1}^m, (w_l)_{l \in L}\right) \in \mathcal{X}$, be such that $v(x_o) = z$, $v(y_{j_k}) = b_{j_k}$ for all $k = 1, ..., m$, $v(w_l) = z_l$

---

[55]Notice that since $\left(z, \sum_{i \in I} \delta_{a_i}, \sum_{l \in L} \delta_{z_l}\right)$ and $\left(z, \sum_{j \in J} \delta_{b_j}, \sum_{l \in L} \delta_{z_l}\right)$ belong to $\mathrm{pid}(u(C))$ we can assume that I, J, L are finite subsets of N with $I \cap L = \emptyset$ and $J \cap L = \emptyset$.

for all $l \in L$. Then $x_o \dot{\succ} y_{j_k}$ for all $k = 1, ..., m$, and m applications of Axiom D.1.ii deliver

$$
\left(x_o, (w_l)_{l \in L} \right) \precsim \left(x_o, y_{j_1}, (w_l)_{l \in L} \right)
$$
$$
\precsim \left(x_o, y_{j_1}, y_{j_2}, (w_l)_{l \in L} \right)
$$
$$
...
$$
$$
\precsim \left(x_o, y_{j_1}, ..., y_{j_m}, (w_l)_{l \in L} \right)
$$

that is $\left(x_o, (w_l)_{l \in L} \right) \precsim \left(x_o, (y_j)_{j \in J}, (w_l)_{l \in L} \right)$ hence

$$
u(x_o) + \varrho \left(v(x_o), \sum_{l \in L} \delta_{v(w_l)} \right) \le u(x_o) + \varrho \left(v(x_o), \sum_{j \in J} \delta_{v(y_j)} + \sum_{l \in L} \delta_{v(w_l)} \right)
$$

and

$$
\eta \left(z, \sum_{i \in I} \delta_{a_i}, \sum_{l \in L} \delta_{z_l} \right) = \varrho \left(z, 0 + \sum_{l \in L} \delta_{z_l} \right) = \varrho \left(v(x_o), \sum_{l \in L} \delta_{v(w_l)} \right)
$$
$$
\le \varrho \left(v(x_o), \sum_{j \in J} \delta_{v(y_j)} + \sum_{l \in L} \delta_{v(w_l)} \right) = \eta \left(z, \sum_{j \in J} \delta_{b_j}, \sum_{l \in L} \delta_{z_l} \right).
$$

If $J = \emptyset$, then $0 \le F_{\mathbf{a}} \le F_{\mathbf{b}} = 0$, and it follows that $I = \emptyset$ (the first case we considered).

Else if $I, J \ne \emptyset$, w.l.o.g. we can assume $I = \{i_1, ..., i_n\}$ and $J = \{j_1, ..., j_m\}$ with $a_{i_1} \le ... \le a_{i_n} < z$ and $b_{j_1} \le ... \le b_{j_m} < z$, and $F_{\mathbf{a}} \le F_{\mathbf{b}}$, by Lemma 5, $n \le m$ and $a_{i_k} \ge b_{j_k}$ for all $k = 1, ..., n$. Let $\left(x_o, (y_{j_k})_{k=1}^{m}, (w_l)_{l \in L} \right), \left(x_o, (x_{i_k})_{k=1}^{n}, (w_l)_{l \in L} \right) \in \mathcal{X}$, be such that $v(x_o) = z$, $v(y_{j_k}) = b_{j_k}$ for all $k = 1, ..., m$, $v(w_l) = z_l$ for all $l \in L$, $v(x_{i_k}) = a_{i_k}$ for all $k = 1, ..., n$. Then $x_o \dot{\succ} x_{i_k} \dot{\succsim} y_{j_k}$ for $k = 1, ..., n$, $x_o \dot{\succ} y_{j_k}$ for $k = n+1, ..., m$, by using n times the definition of $\dot{\succsim}$ and $m - n$ times Axiom D.1.ii we obtain

$$
\left(x_o, x_{j_1}, x_{j_2}, ..., x_{j_n}, (w_l)_{l \in L} \right) \precsim \left(x_o, y_{j_1}, x_{j_2}, ..., x_{j_n}, (w_l)_{l \in L} \right)
$$
$$
...
$$
$$
\precsim \left(x_o, y_{j_1}, ..., y_{j_n}, (w_l)_{l \in L} \right)
$$
$$
\precsim \left(x_o, y_{j_1}, ..., y_{j_n}, y_{j_{n+1}}, (w_l)_{l \in L} \right)
$$
$$
...
$$
$$
\precsim \left(x_o, y_{j_1}, ..., y_{j_n}, ..., y_{j_m}, (w_l)_{l \in L} \right)
$$

that is $\left(x_o, (x_{j_k})_{k=1}^{n}, (w_l)_{l \in L} \right) \precsim \left(x_o, (y_{j_k})_{k=1}^{m}, (w_l)_{l \in L} \right)$ hence

$$
u(x_o) + \varrho \left(v(x_o), \sum_{k=1}^{n} \delta_{v(x_{j_k})} + \sum_{l \in L} \delta_{v(w_l)} \right) \le u(x_o) + \varrho \left(v(x_o), \sum_{k=1}^{m} \delta_{v(y_{j_k})} + \sum_{l \in L} \delta_{v(w_l)} \right)
$$

and

$$
\eta \left(z, \sum_{i \in I} \delta_{a_i}, \sum_{l \in L} \delta_{z_l} \right) = \varrho \left(z, \sum_{k=1}^{n} \delta_{a_{i_k}} + \sum_{l \in L} \delta_{z_l} \right) = \varrho \left(v(x_o), \sum_{k=1}^{n} \delta_{v(x_{j_k})} + \sum_{l \in L} \delta_{v(w_l)} \right)
$$
$$
\le \varrho \left(v(x_o), \sum_{k=1}^{m} \delta_{v(y_{j_k})} + \sum_{l \in L} \delta_{v(w_l)} \right) = \eta \left(z, \sum_{j \in J} \delta_{b_j}, \sum_{l \in L} \delta_{z_l} \right).
$$

Monotonicity of η in the third component w.r.t. the upper dominance is proved in the same way.

In fact, let $\left(z, \sum_{l \in L} \delta_{z_l}, \sum_{i \in I} \delta_{a_i}\right), \left(z, \sum_{l \in L} \delta_{z_l}, \sum_{j \in J} \delta_{b_j}\right) \in \mathrm{pid}\,(v\,(C))$ and assume $\sum_{i \in I} \delta_{a_i}$ upper dominates $\sum_{j \in J} \delta_{b_j}$.[56]

If $J = \emptyset$ and $I = \emptyset$, then

$$\eta\left(z, \sum_{l \in L} \delta_{z_l}, \sum_{i \in I} \delta_{a_i}\right) = \eta\left(z, \sum_{l \in L} \delta_{z_l}, 0\right) = \eta\left(z, \sum_{l \in L} \delta_{z_l}, \sum_{j \in J} \delta_{b_j}\right)$$

If $J = \emptyset$ and $I \neq \emptyset$, then w.l.o.g. we can assume $I = \{i_1, ..., i_n\}$ with $z \leq a_{i_n} \leq a_{i_{n-1}} \leq ... \leq a_{i_1}$. Let $\left(x_o, (w_l)_{l \in L}, (x_{i_k})_{k=1}^n\right) \in \mathcal{X}$, be such that $v\,(x_o) = z$, $v\,(w_l) = z_l$ for all $l \in L$, $v\,(x_{i_k}) = a_{i_k}$ for all $k = 1, ..., n$. Then $x_{i_k} \dot{\succsim} x_o$ for $k = 1, ..., n$, applying n times Axiom D.1.i, we obtain

$$\left(x_o, (w_l)_{l \in L}\right) \succsim \left(x_o, (w_l)_{l \in L}, x_{i_1}\right)$$
$$\succsim \left(x_o, (w_l)_{l \in L}, x_{i_1}, x_{i_2}\right)$$
$$...$$
$$\succsim \left(x_o, (w_l)_{l \in L}, x_{i_1}, ..., x_{i_n}\right)$$

that is $\left(x_o, (w_l)_{l \in L}, (x_i)_{i \in I}\right) \precsim \left(x_o, (w_l)_{l \in L}\right)$ hence

$$\eta\left(z, \sum_{l \in L} \delta_{z_l}, \sum_{i \in I} \delta_{a_i}\right) = \varrho\left(z, \sum_{l \in L} \delta_{z_l} + \sum_{i \in I} \delta_{a_i}\right) = \varrho\left(v\,(x_o), \sum_{l \in L} \delta_{v(w_l)} + \sum_{i \in I} \delta_{v(x_i)}\right)$$

$$\leq \varrho\left(v\,(x_o), \sum_{l \in L} \delta_{v(w_l)}\right) = \eta\left(z, \sum_{l \in L} \delta_{z_l}, \sum_{j \in J} \delta_{b_j}\right)$$

If $I = \emptyset$, then $0 \leq G_{\mathbf{b}} \leq G_{\mathbf{a}} = 0$, and it follows that $J = \emptyset$ (the first case we considered).

Else if $I, J \neq \emptyset$, w.l.o.g. $I = \{i_1, ..., i_n\}$ and $J = \{j_1, ..., j_m\}$ with $z \leq a_{i_n} \leq ... \leq a_{i_1}$ and $z \leq b_{j_m} \leq ... \leq b_{j_1}$, and $G_{\mathbf{a}} \geq G_{\mathbf{b}}$, by Lemma 6, $n \geq m$ and $a_{i_k} \geq b_{j_k}$ for all $k = 1, ..., m$. Let $\left(x_o, (w_l)_{l \in L}, (x_{i_k})_{k=1}^n\right), \left(x_o, (w_l)_{l \in L}, (y_{j_k})_{k=1}^m\right) \in \mathcal{X}$, be such that $v\,(x_o) = z$, $v\,(x_{i_k}) = a_{i_k}$ for all $k = 1, ..., n$, $v\,(w_l) = z_l$ for all $l \in L$, $v\,(y_{j_k}) = b_{j_k}$ for all $k = 1, ..., m$. Then $x_{i_k} \dot{\succsim} y_{j_k} \dot{\succsim} x_o$ for $k = 1, ..., m$, $x_{i_k} \dot{\succsim} x_o$ for $k = m+1, ..., n$, by using m times the definition of $\dot{\succsim}$ and $n - m$ times Axiom D.1.i we obtain

$$\left(x_o, (w_l)_{l \in L}, y_{i_1}, y_{i_2}, ..., y_{i_m}\right) \succsim \left(x_o, (w_l)_{l \in L}, x_{i_1}, y_{i_2}, ..., y_{i_m}\right)$$
$$...$$
$$\succsim \left(x_o, (w_l)_{l \in L}, x_{i_1}, ..., x_{i_m}\right)$$
$$\succsim \left(x_o, (w_l)_{l \in L}, x_{i_1}, ..., x_{i_m}, x_{i_{m+1}}\right)$$
$$...$$
$$\succsim \left(x_o, (w_l)_{l \in L}, x_{i_1}, ..., x_{i_m}, ..., x_{i_n}\right)$$

---

[56]Notice that since $\left(z, \sum_{l \in L} \delta_{z_l}, \sum_{i \in I} \delta_{a_i}\right)$ and $\left(z, \sum_{l \in L} \delta_{z_l}, \sum_{j \in J} \delta_{b_j}\right)$ belong to $\mathrm{pid}\,(v\,(C))$ we can assume that I, J, L are finite subsets of N with $I \cap L = \emptyset$ and $J \cap L = \emptyset$.

that is $(x_o, (w_l)_{l \in L}, (x_{i_k})_{k=1}^n) \precsim (x_o, (w_l)_{l \in L}, (y_{i_k})_{k=1}^m)$ hence

$$\eta \left(z, \sum_{l \in L} \delta_{z_l}, \sum_{i \in I} \delta_{a_i} \right) = \varrho \left(z, \sum_{l \in L} \delta_{z_l} + \sum_{k=1}^n \delta_{a_{i_k}} \right) = \varrho \left(v(x_o), \sum_{l \in L} \delta_{v(w_l)} + \sum_{k=1}^n \delta_{v(x_{i_k})} \right)$$

$$\leq \varrho \left(v(x_o), \sum_{l \in L} \delta_{v(w_l)} + \sum_{k=1}^m \delta_{v(y_{i_k})} \right) = \eta \left(z, \sum_{l \in L} \delta_{z_l}, \sum_{j \in J} \delta_{b_j} \right).$$

The proof of sufficiency is concluded by observing that for all $\mu = \sum_{i \in I} \delta_{a_i} \in \mathcal{M}(v(C))$ and all $z \in v(C)$,

$$\varrho(z, \mu) = \varrho \left(z, \sum_{i:a_i < z} \delta_{a_i} + \sum_{i:a_i \geq z} \delta_{a_i} \right) = \eta \left(z, \sum_{i:a_i < z} \delta_{a_i}, \sum_{i:a_i \geq z} \delta_{a_i} \right)$$

and so

$$V(f_o, (f_i)_{i \in I}) = \int_S \left[u(f_o(s)) + \varrho \left(v(f_o(s)), \sum_{i \in I} \delta_{v(f_i(s))} \right) \right] dP(s)$$

$$= \int_S \left[u(f_o(s)) + \eta \left(v(f_o(s)), \sum_{i:v(f_i(s)) < v(f_o(s))} \delta_{v(f_i(s))}, \sum_{i:v(f_i(s)) \geq v(f_o(s))} \delta_{v(f_i(s))} \right) \right] dP(s).$$

for all $(f_o, (f_i)_{i \in I}) \in \mathcal{F}$.

Conversely, assume that there exist two non-constant affine functions $u, v : C \to \mathbb{R}$, a diago-null function $\eta : \mathrm{pid}(v(C)) \to \mathbb{R}$ increasing in the first component, decreasing in the second and third components w.r.t. lower dominance and upper dominance respectively, and a probability P on Σ such that the functional $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V(f_0, (f_i)_{i \in I}) = \int_S u(f_o(s)) dP(s)$$

$$+ \int_S \eta \left(v(f_o(s)), \sum_{i:v(f_i(s)) < v(f_o(s))} \delta_{v(f_i(s))}, \sum_{i:v(f_i(s)) \geq v(f_o(s))} \delta_{v(f_i(s))} \right) dP(s)$$

represents $\succsim$ on $\mathcal{F}$ and satisfies $V(\mathcal{F}) = u(C)$.

For all $(z, \mu) \in \mathrm{pim}(v(C))$, $\sum_{r \in \mathrm{supp}(\mu):r<z} \mu(r) \delta_r$ and $\sum_{r \in \mathrm{supp}(\mu):r \geq z} \mu(r) \delta_r$ are positive integer measures finitely supported in $v(C) \cap (-\infty, z)$ and $v(C) \cap [z, \infty)$ respectively, and their sum μ has total mass bounded by $|N|$, that is

$$\left(z, \sum_{r \in \mathrm{supp}(\mu):r<z} \mu(r) \delta_r, \sum_{r \in \mathrm{supp}(\mu):r \geq z} \mu(r) \delta_r \right) \in \mathrm{pid}(v(C)).$$

Define

$$\varrho(z, \mu) = \eta \left(z, \sum_{r \in \mathrm{supp}(\mu):r<z} \mu(r) \delta_r, \sum_{r \in \mathrm{supp}(\mu):r \geq z} \mu(r) \delta_r \right)$$

for all $(z, \mu) \in \mathrm{pim}(v(C))$. Notice that $\varrho(z, n\delta_z) = \eta(z, 0, n\delta_z) = 0$ for all z in $v(C)$ and all non-negative integers $n \leq |N|$, that is ϱ is diago-null. Moreover, ϱ is increasing in its first component (since η is).

Let $z \in v(C)$ and $\mu, \mu' \in \mathcal{M}(C)$ be such that μ stochastically dominates μ'. If $\mu = \mu' = 0$, then $\varrho(z, \mu) = \varrho(z, \mu') = 0$. Else there exist $n \in \mathbb{N}$ $(n > 0)$ and $(a_i)_{i=1}^n, (b_i)_{i=1}^n \in v(C)^n$ such

that $a_i \geq b_i$ for all $i = 1, ..., n$, $\mu = \sum_{i=1}^{n} \delta_{a_i}$, and $\mu' = \sum_{i=1}^{n} \delta_{b_i}$. By Lemma 7, $G_{(a_i)_{i:a_i \geq z}}(t) \geq G_{(b_j)_{j:b_j \geq z}}(t)$ for all $t \in \mathbb{R}$ and $F_{(a_i)_{i:a_i < z}}(t) \leq F_{(b_j)_{j:b_j < z}}(t)$ for all $t \in \mathbb{R}$, therefore

$$
\begin{aligned}
\varrho(z, \mu) &= \eta \left(z, \sum_{a \in \text{supp}(\mu): a < z} \mu(a) \delta_a, \sum_{a \in \text{supp}(\mu): a \geq z} \mu(a) \delta_a \right) \\
&= \eta \left(z, \sum_{i: a_i < z} \delta_{a_i}, \sum_{i: a_i \geq z} \delta_{a_i} \right) \\
&\leq \eta \left(z, \sum_{i: b_i < z} \delta_{b_i}, \sum_{i: b_i \geq z} \delta_{b_i} \right) = \varrho(z, \mu').
\end{aligned}
$$

We conclude that $\varrho : \text{pim}(v(C)) \to \mathbb{R}$ is decreasing w.r.t. stochastic dominance in the second component. By Theorem 2, $\succsim$ on $\mathcal{F}$ satisfies Axioms A.1-A.11. It remains to show that $\succsim$ satisfies Axiom D.1.

Let $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$, $j \in N \setminus I$, and $x_j \in C$. If $x_j \dot{\succsim} x_o$, then $v(x_j) \geq v(x_o)$ and, for all $t \in \mathbb{R}$,

$$
\begin{aligned}
G_{v(x_i)_{i \in I_j : v(x_i) \geq v(x_o)}}(t) &= \sum_{i \in I_j : v(x_i) \geq v(x_o)} \delta_{v(x_i)}(t, \infty) = \sum_{i \in I : v(x_i) \geq v(x_o)} \delta_{v(x_i)}(t, \infty) + \delta_{v(x_j)}(t, \infty) \\
&\geq G_{v(x_i)_{i \in I : v(x_i) \geq v(x_o)}}(t)
\end{aligned}
$$

that is $\sum_{i \in I_j : v(x_i) \geq v(x_o)} \delta_{v(x_i)}$ upper stochastically dominates $\sum_{i \in I : v(x_i) \geq v(x_o)} \delta_{v(x_i)}$, while, at the same time, $\sum_{i \in I_j : v(x_i) < v(x_o)} \delta_{v(x_i)} = \sum_{i \in I : v(x_i) < v(x_o)} \delta_{v(x_i)}$, monotonicity of η implies

$$
\begin{aligned}
\eta \left(v(x_o), \sum_{i \in I : v(x_i) < v(x_o)} \delta_{v(x_i)}, \sum_{i \in I : v(x_i) \geq v(x_o)} \delta_{v(x_i)} \right) &\geq \\
\geq \eta \left(v(x_o), \sum_{i \in I_j : v(x_i) < v(x_o)} \delta_{v(x_i)}, \sum_{i \in I_j : v(x_i) \geq v(x_o)} \delta_{v(x_i)} \right), &\quad \text{then} \\
u(x_o) + \eta \left(v(x_o), \sum_{i \in I : v(x_i) < v(x_o)} \delta_{v(x_i)}, \sum_{i \in I : v(x_i) \geq v(x_o)} \delta_{v(x_i)} \right) &\geq \\
\geq u(x_o) + \eta \left(v(x_o), \sum_{i \in I_j : v(x_i) < v(x_o)} \delta_{v(x_i)}, \sum_{i \in I_j : v(x_i) \geq v(x_o)} \delta_{v(x_i)} \right) &\quad \text{and} \\
\left(x_o, (x_i)_{i \in I} \right) \succsim \left(x_o, (x_i)_{i \in I_j} \right)
\end{aligned}
$$

Analogously, if $x_j \dot{\prec} x_o$, then $v(x_j) < v(x_o)$ and, for all $t \in \mathbb{R}$,

$$
\begin{aligned}
F_{v(x_i)_{i \in I_j : v(x_i) < v(x_o)}}(t) &= \sum_{i \in I_j : v(x_i) < v(x_o)} \delta_{v(x_i)}(-\infty, t] \\
&= \sum_{i \in I : v(x_i) < v(x_o)} \delta_{v(x_i)}(-\infty, t] + \delta_{v(x_j)}(-\infty, t] \geq F_{v(x_i)_{i \in I : v(x_i) < v(x_o)}}(t)
\end{aligned}
$$

that is $\sum_{i \in I : v(x_i) < v(x_o)} \delta_{v(x_i)}$ upper stochastically dominates $\sum_{i \in I_j : v(x_i) \geq v(x_o)} \delta_{v(x_i)}$, while, at the

same time, $\sum_{i\in I:v(x_i)\geq v(x_o)}\delta_{v(x_i)}=\sum_{i\in I_j:v(x_i)\geq v(x_o)}\delta_{v(x_i)}$, monotonicity of η implies

$$\eta\left(v\left(x_o\right),\sum_{i\in I:v(x_i)<v(x_o)}\delta_{v(x_i)},\sum_{i\in I:v(x_i)\geq v(x_o)}\delta_{v(x_i)}\right)\leq$$

$$\leq\eta\left(v\left(x_o\right),\sum_{i\in I_j:v(x_i)<v(x_o)}\delta_{v(x_i)},\sum_{i\in I_j:v(x_i)\geq v(x_o)}\delta_{v(x_i)}\right),\text{ then}$$

$$u\left(x_o\right)+\eta\left(v\left(x_o\right),\sum_{i\in I:v(x_i)<v(x_o)}\delta_{v(x_i)},\sum_{i\in I:v(x_i)\geq v(x_o)}\delta_{v(x_i)}\right)\leq$$

$$\leq u\left(x_o\right)+\eta\left(v\left(x_o\right),\sum_{i\in I_j:v(x_i)<v(x_o)}\delta_{v(x_i)},\sum_{i\in I_j:v(x_i)\geq v(x_o)}\delta_{v(x_i)}\right)\text{ and}$$

$$\left(x_o,(x_i)_{i\in I}\right)\precsim\left(x_o,(x_i)_{i\in I_j}\right)$$

This completes the proof. ∎

Proof of Proposition 6. Let $\left(\hat{u},\hat{v},\hat{\varrho},\hat{P}\right)$ be another representation of $\succsim$ and $\overset{\cdot}{\succsim}$ in the sense of Theorem 5. Set $r\left(x_o,(x_i)_{i\in I}\right)=\varrho\left(v\left(x_o\right),\sum_{i\in I:v(x_i)<v(x_o)}\delta_{v(x_i)},\sum_{i\in I:v(x_i)\geq v(x_o)}\delta_{v(x_i)}\right)$ and $\hat{r}\left(x_o,(x_i)_{i\in I}\right)=\hat{\varrho}\left(\hat{v}\left(x_o\right),\sum_{i\in I:\hat{v}(x_i)<\hat{v}(x_o)}\delta_{\hat{v}(x_i)},\sum_{i\in I:\hat{v}(x_i)\geq\hat{v}(x_o)}\delta_{\hat{v}(x_i)}\right)$ for all $\left(x_o,(x_i)_{i\in I}\right)\in\mathcal{X}$. By Lemma 9, there exist $\alpha,\beta\in\mathbb{R}$ with $\alpha>0$ such that $\hat{u}=\alpha u+\beta$, $\hat{r}=\alpha r$, and $\hat{P}=P$. Moreover, since $\hat{v}$ represents $\overset{\cdot}{\succsim}$, there are $\dot{\alpha},\dot{\beta}\in\mathbb{R}$ with $\dot{\alpha}>0$ such that $\hat{v}=\dot{\alpha}v+\dot{\beta}$. Let $\left(z,\sum_{i\in I}\delta_{z_i},\sum_{j\in J}\delta_{r_j}\right)\in\mathrm{pid}\left(\hat{v}\left(C\right)\right)$, then there exist $x=\left(x_o,(x_i)_{i\in I},(y_j)_{j\in J}\right)\in\mathcal{X}$ such that $\left(z,(z_i)_{i\in I},(r_j)_{j\in J}\right)=\left(\hat{v}\left(x_o\right),(\hat{v}\left(x_i\right))_{i\in I},(\hat{v}\left(y_j\right))_{j\in J}\right)$. Therefore $\left(z,\sum_{i\in I}\delta_{z_i},\sum_{j\in J}\delta_{r_j}\right)=\left(\hat{v}\left(x_o\right),\sum_{i\in I}\delta_{\hat{v}(x_i)},\sum_{j\in J}\delta_{\hat{v}(y_j)}\right)$, and from $\hat{r}=\alpha r$ it follows that

$$\hat{\varrho}\left(z,\sum_{i\in I}\delta_{z_i},\sum_{j\in J}\delta_{r_j}\right)=\hat{\varrho}\left(\hat{v}\left(x_o\right),\sum_{i\in I}\delta_{\hat{v}(x_i)},\sum_{j\in J}\delta_{\hat{v}(y_j)}\right)=\hat{r}\left(x_o,(x_i)_{i\in I},(y_j)_{j\in J}\right)$$

$$=\alpha r\left(x_o,(x_i)_{i\in I},(y_j)_{j\in J}\right)=\alpha\varrho\left(v\left(x_o\right),\sum_{i\in I}\delta_{v(x_i)},\sum_{j\in J}\delta_{v(y_j)}\right)$$

$$=\alpha\varrho\left(\frac{z-\dot{\beta}}{\dot{\alpha}},\sum_{i\in I}\delta_{\frac{z_i-\dot{\beta}}{\dot{\alpha}}},\sum_{j\in J}\delta_{\frac{r_j-\dot{\beta}}{\dot{\alpha}}}\right)$$

since $\hat{v}=\dot{\alpha}v+\dot{\beta}$ amounts to $v=\dot{\alpha}^{-1}\left(\hat{v}-\dot{\beta}\right)$.

Conversely, if $\hat{P}=P$, and there exist $\alpha,\beta,\dot{\alpha},\dot{\beta}\in\mathbb{R}$ with $\alpha,\dot{\alpha}>0$ such that $\hat{u}=\alpha u+\beta$, $\hat{v}=\dot{\alpha}v+\dot{\beta}$, and $\hat{\varrho}\left(z,\sum_{i\in I}\delta_{z_i},\sum_{j\in J}\delta_{r_j}\right)=\alpha\varrho\left(\frac{z-\dot{\beta}}{\dot{\alpha}},\sum_{i\in I}\delta_{\frac{z_i-\dot{\beta}}{\dot{\alpha}}},\sum_{j\in J}\delta_{\frac{r_j-\dot{\beta}}{\dot{\alpha}}}\right)$ for all $\left(z,\sum_{i\in I}\delta_{z_i},\sum_{j\in J}\delta_{r_j}\right)\in$ $\mathrm{pid}\left(\hat{v}\left(C\right)\right)$, then $\hat{u},\hat{v}:C\to\mathbb{R}$ are non-constant affine, it is easy to check that $\hat{\varrho}:\mathrm{pid}\left(\hat{v}\left(C\right)\right)\to\mathbb{R}$ is well defined, diago-null, increasing in the first component and decreasing in the second and third components w.r.t. lower dominance and upper dominance respectively, $\hat{P}$ is a probability

on Σ, $\hat{v}$ represents $\overset{\cdot}{\underset{\sim}{\succeq}}$, and

$$\hat{V}(f) = \int_S \left[\hat{u}(f_o(s)) + \hat{\varrho}\left(\hat{v}(f_o(s)), \sum_{i \in I : \hat{v}(f_i(s)) < \hat{v}(f_o(s))} \delta_{\hat{v}(f_i(s))}, \sum_{i \in I : \hat{v}(f_i(s)) \geq \hat{v}(f_o(s))} \delta_{\hat{v}(f_i(s))} \right) \right] dP(s)$$

$$= \int_S \left[\alpha u(f_o(s)) + \beta + \alpha\varrho\left(\frac{\hat{v}(f_o(s)) - \dot{\beta}}{\dot{\alpha}}, \sum_{i \in I : v(f_i(s)) < v(f_o(s))} \delta_{\frac{\hat{v}(f_i(s)) - \dot{\beta}}{\dot{\alpha}}}, \sum_{i \in I : v(f_i(s)) \geq v(f_o(s))} \delta_{\frac{\hat{v}(f_i(s)) - \dot{\beta}}{\dot{\alpha}}} \right) \right] dP(s)$$

$$= \alpha V(f) + \beta$$

obviously represents $\succsim$ on $\mathcal{F}$; finally $\hat{V}(\mathcal{F}) = \alpha V(\mathcal{F}) + \beta = \alpha u(C) + \beta = \hat{u}(C)$. ∎

Proof of Proposition 7. First, observe that for a real valued function ϕ defined on an interval $K \ni z$ the following statements are equivalent:

(i) $\phi(z) \geq \phi(z+h) + \phi(z-h)$ for all $h \geq 0$ such that $z \pm h \in K$,

(ii) $\phi(z) \geq \phi(t) + \phi(w)$ for all $t, w \in K$ such that $t/2 + w/2 = z$.[57]

Assume $\succsim$ is more envious than proud, relative to an ethically neutral event E, a convex $D \subseteq C$, and $x_o \in D$. Let $t, w \in v(D)$ be such that $t/2 + w/2 = v(x_o)$. Choose $x_i, y_i \in D$ such that $t = v(x_i)$ and $w = v(y_i)$. Then

$$v\left(\frac{1}{2}x_i + \frac{1}{2}y_i \right) = \frac{v(x_i)}{2} + \frac{v(y_i)}{2} = v(x_o)$$

implies $(1/2)x_i + (1/2)y_i \overset{\cdot}{\sim} x_o$, and the assumption of social loss aversion delivers

$(x_o, x_o) \succsim (x_o, x_i E y_i)$

$\Longrightarrow u(x_o) \geq P(E)(u(x_o) + \varrho(v(x_o), v(x_i))) + (1 - P(E))(u(x_o) + \varrho(v(x_o), v(y_i)))$

$\Longrightarrow u(x_o) \geq \frac{1}{2}(u(x_o) + \varrho(v(x_o), v(x_i))) + \frac{1}{2}(u(x_o) + \varrho(v(x_o), v(y_i)))$

$\Longrightarrow 0 \geq \varrho(v(x_o), v(x_i)) + \varrho(v(x_o), v(y_i))$

$\Longrightarrow \varrho(v(x_o), v(x_o)) \geq \varrho(v(x_o), t) + \varrho(v(x_o), w).$

Therefore $0 = \varrho(v(x_o), v(x_o)) \geq \varrho(v(x_o), v(x_o) + h) + \varrho(v(x_o), v(x_o) - h)$ for all $h \geq 0$ such that $v(x_o) \pm h \in v(D)$.

Conversely, if (15) holds, then

$$\varrho(v(x_o), v(x_o) + h) + \varrho(v(x_o), v(x_o) - h) \leq 0 = \varrho(v(x_o), v(x_o))$$

for all $h \geq 0$ such that $z \pm h \in v(D)$, that is

$$\varrho(v(x_o), v(x_o)) \geq \varrho(v(x_o), t) + \varrho(v(x_o), w)$$

for all $t, w \in v(D)$ such that $t/2 + w/2 = v(x_o)$. If $x_i, y_i \in D$ are such that $(1/2)x_i + (1/2)y_i \overset{\cdot}{\sim} x_o$, then

$$\frac{v(x_i)}{2} + \frac{v(y_i)}{2} = v\left(\frac{1}{2}x_i + \frac{1}{2}y_i \right) = v(x_o)$$

---

[57](i)$\Rightarrow$(ii) If $t, w \in K$ are such that $t/2 + w/2 = z$, and $t \geq w$, set $h = (t - w)/2$, it follows that $h \geq 0$ and that $z + h = t/2 + w/2 + (t/2 - w/2) = t \in K$, $z - h = t/2 + w/2 - (t/2 - w/2) = w \in K$. By (i), $\phi(z) \geq \phi(z+h) + \phi(z-h) = \phi(t) + \phi(w)$. If $t \leq w$, set $h = (w - t)/2$ and repeat the same argument.

(ii)$\Rightarrow$(i) If $h \geq 0$ is such that $z \pm h \in K$, then, from $(z+h)/2 + (z-h)/2 = z$ and (ii), it follows that $\phi(z) \geq \phi(z+h) + \phi(z-h)$.

and hence

$$0 = \varrho(v(x_o), v(x_o)) \geq \varrho(v(x_o), v(x_i)) + \varrho(v(x_o), v(y_i))$$

$$\implies 0 \geq \frac{1}{2}\varrho(v(x_o), v(x_i)) + \frac{1}{2}\varrho(v(x_o), v(y_i))$$

$$\implies u(x_o) \geq \frac{1}{2}u(x_o) + \frac{1}{2}\varrho(v(x_o), v(x_i)) + \frac{1}{2}u(x_o) + \frac{1}{2}\varrho(v(x_o), v(y_i))$$

$$\implies u(x_o) \geq P(E)(u(x_o) + \varrho(v(x_o), v(x_i))) + (1 - P(E))(u(x_o) + \varrho(v(x_o), v(y_i)))$$

$$\implies (x_o, x_o) \succsim (x_o, x_i E y_i).$$

Thus $\succsim$ is more envious than proud. Finally, inequality (16) easily follows from (15). In fact, let $v(x_o) = r \in \text{int}(v(D))$, since $\varrho(r, r) = 0$

$$\underline{D}_2^+ \varrho(r, r) = \liminf_{h \downarrow 0} \frac{\varrho(r, r + h) - \varrho(r, r)}{h} = \lim_{\varepsilon \downarrow 0} \inf_{h \in (0, \varepsilon)} \frac{\varrho(r, r + h)}{h}$$

$$\leq \lim_{\varepsilon \downarrow 0} \inf_{h \in (0, \varepsilon)} \frac{-\varrho(r, r - h)}{h} = \lim_{\varepsilon \downarrow 0} \inf_{h \in (0, \varepsilon)} \frac{\varrho(r, r - h)}{-h}$$

$$= \lim_{\varepsilon \downarrow 0} \inf_{h \in (-\varepsilon, 0)} \frac{\varrho(r, r + h)}{h} \leq \lim_{\varepsilon \downarrow 0} \inf_{h \in (-\varepsilon, 0)} \frac{\varrho(r, r + h) - \varrho(r, r)}{h}$$

$$= \liminf_{h \uparrow 0} \frac{\varrho(r, r + h) - \varrho(r, r)}{h} = \underline{D}_2^- \varrho(r, r).$$

as wanted. ∎

Before entering the details of the proof of Proposition 8, recall that an event $E \in \Sigma$ is *essential* if $\bar{c} \nsim \bar{c} E \underline{c} \nsim \underline{c}$ for some $\bar{c}$ and $\underline{c}$ in C. Representation (11) guarantees that this amounts to say that $P(E) \in (0, 1)$, in particular, ethically neutral events are essential.

We say that a preference $\succsim$ is *averse to social risk*, relatively to an essential event E, a convex set $D \subseteq C$, and a given $x_o \in C$, if

$$(x_o, w_i) \succsim (x_o, x_i E y_i)$$

for all $x_i, y_i, w_i \in D$ such that $P(E) x_i + (1 - P(E)) y_i \sim w_i$. Notice that this definition is consistent with the previous one in which only ethically neutral events E where considered (thus $P(E) = 1/2$). Instead of **proving Proposition 8** we will prove the more general

Proposition 15 *If $\succsim$ admits a representation (11), then $\succsim$ is averse to social risk, relative to an essential event E, a convex $D \subseteq C$, and $x_o \in C$ if and only if $\varrho(v(x_o), \cdot)$ is convex on $v(D)$.*

Proof of Proposition 15. Assume $\succsim$ is averse to social risk, relative to an essential event E, a convex $D \subseteq C$, and $x_o \in C$. Essentiality of E guarantees that $P(E) = p \in (0, 1)$. Therefore, for all $t = v(x_i), r = v(y_i) \in v(D)$, social risk aversion implies $(x_o, px_i + (1 - p) y_i) \succsim (x_o, x_i E y_i)$ and

$$u(x_o) + \varrho(v(x_o), v(px_i + (1 - p) y_i)) \geq p(u(x_o) + \varrho(v(x_o), v(x_i))) + (1 - p)(u(x_o) + \varrho(v(x_o), v(y_i)))$$

$$u(x_o) + \varrho(v(x_o), pt + (1 - p) r) \geq u(x_o) + p\varrho(v(x_o), t) + (1 - p)\varrho(v(x_o), r)$$

$$\varrho(v(x_o), pt + (1 - p) r) \geq p\varrho(v(x_o), t) + (1 - p)\varrho(v(x_o), r)$$

In turn, this (together with monotonicity of ϱ in the second component) can be shown to imply continuity of $\varrho(v(x_o), \cdot)$ on $v(D) \setminus \sup v(D)$. Theorem 88 of Hardy, Littlewood and Polya (1934) guarantees concavity of $\varrho(v(x_o), \cdot)$ on $v(D) \setminus \sup v(D)$. Monotonicity again delivers concavity of $\varrho(v(x_o), \cdot)$ on $v(D)$.

The converse is trivial. ∎

Proof of Proposition 9. (i)⇒(ii) Taking $I = \emptyset$, since u_1 and u_2 are affine, non-constant, and represent $\succsim_1$ and $\succsim_2$ on C, we obtain $u_1 \approx u_2$. W.l.o.g. choose $u_1 = u_2 = u$. For all $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$ choose $c \in C$ such that $\left(x_o, (x_i)_{i \in I}\right) \sim_1 c$, then $\left(x_o, (x_i)_{i \in I}\right) \succsim_2 c$ and

$$u(x_o) + \varrho_2\left(x_o, \mu_{(x)_{i \in I}}\right) \geq u(c) = u(x_o) + \varrho_1\left(x_o, \mu_{(x)_{i \in I}}\right),$$

that is $\varrho_2 \geq \varrho_2$ on $\text{pim}(C)$.

(ii)⇒(i) Take $u_1 = u_2 = u$. If $\left(x_o, (x_i)_{i \in I}\right) \succsim_1 c_{I_o}$, then $u(x_o) + \varrho_1\left(x_o, \mu_{(x)_{i \in I}}\right) \geq u(c)$ hence $u(x_o) + \varrho_2\left(x_o, \mu_{(x)_{i \in I}}\right) \geq u(c)$ and $\left(x_o, (x_i)_{i \in I}\right) \succsim_2 c_{I_o}$. As wanted. ∎

Proof of Proposition 10. By Proposition 9, (i) is equivalent to $u_1 \approx u_2$ and, choosing $u_1 = u_2$, $\varrho_1 \leq \varrho_2$.

(i)⇒(ii) Intrinsic equivalence is obvious. Take $u_1 = u_2 = u$. If $x_o \succ_2 y_o \succsim_2 \left(x_o, (x_i)_{i \in I}\right)$, then

$$u(x_o) > u(y_o) \geq u(x_o) + \varrho_2\left(x_o, \mu_{(x)_{i \in I}}\right)$$
$$\varrho_2\left(x_o, \mu_{(x)_{i \in I}}\right) \leq u(y_o) - u(x_o) < 0$$
$$\varrho_1\left(x_o, \mu_{(x)_{i \in I}}\right) \leq u(y_o) - u(x_o) < 0$$
$$u(x_o) + \varrho_1\left(x_o, \mu_{(x)_{i \in I}}\right) \leq u(y_o) < u(x_o)$$
$$x_o \succ_1 y_o \succsim_1 \left(x_o, (x_i)_{i \in I}\right).$$

Analogously, if $x_o \prec_1 y_o \precsim_1 \left(x_o, (x_i)_{i \in I}\right)$,

$$u(x_o) < u(y_o) \leq u(x_o) + \varrho_1\left(x_o, \mu_{(x)_{i \in I}}\right)$$
$$0 < u(y_o) - u(x_o) \leq \varrho_1\left(x_o, \mu_{(x)_{i \in I}}\right)$$
$$0 < u(y_o) - u(x_o) \leq \varrho_2\left(x_o, \mu_{(x)_{i \in I}}\right)$$
$$u(x_o) < u(y_o) \leq u(x_o) + \varrho_2\left(x_o, \mu_{(x)_{i \in I}}\right)$$
$$x_o \prec_2 y_o \precsim_2 \left(x_o, (x_i)_{i \in I}\right).$$

(ii)⇒(i) Since u_1 and u_2 are affine, if $\succsim_1$ is intrinsically equivalent to $\succsim_2$, then $u_1 \approx u_2$. W.l.o.g. choose $u_1 = u_2 = u$. For all $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$ choose c such that $c \sim_2 \left(x_o, (x_i)_{i \in I}\right)$, i.e. $u(c) = u(x_o) + \varrho_2\left(x_o, \mu_{(x)_{i \in I}}\right)$, and $\bar{c}$ such that $\bar{c} \sim_1 \left(x_o, (x_i)_{i \in I}\right)$.

If $\varrho_2\left(x_o, \mu_{(x)_{i \in I}}\right) < 0$,

$$u(x_o) + \varrho_2\left(x_o, \mu_{(x)_{i \in I}}\right) < u(x_o)$$
$$x_o \succ_2 c \sim_2 \left(x_o, (x_i)_{i \in I}\right)$$
$$x_o \succ_1 c \succsim_1 \left(x_o, (x_i)_{i \in I}\right)$$
$$u(x_o) + \varrho_1\left(x_o, \mu_{(x)_{i \in I}}\right) \leq u(c) < u(x_o)$$

then $\varrho_1\left(x_o, \mu_{(x)_{i\in I}}\right) \le u(c) - u(x_o) = \varrho_2\left(x_o, \mu_{(x)_{i\in I}}\right)$. Analogously, if $\varrho_1\left(x_o, \mu_{(x)_{i\in I}}\right) > 0$,

$$u(x_o) + \varrho_1\left(x_o, \mu_{(x)_{i\in I}}\right) > u(x_o)$$
$$x_o \prec_1 \bar{c} \sim_1 \left(x_o, (x_i)_{i\in I}\right)$$
$$x_o \prec_2 \bar{c} \precsim_2 \left(x_o, (x_i)_{i\in I}\right)$$
$$u(x_o) + \varrho_2\left(x_o, \mu_{(x)_{i\in I}}\right) \ge u(\bar{c}) > u(x_o)$$

then $\varrho_2\left(x_o, \mu_{(x)_{i\in I}}\right) \ge u(\bar{c}) - u(x_o) = \varrho_1\left(x_o, \mu_{(x)_{i\in I}}\right)$. Conclude that: if $\varrho_2(\cdot) < 0$ then $\varrho_1(\cdot) \le \varrho_2(\cdot)$, if $\varrho_2(\cdot) \ge 0$, then either $\varrho_1(\cdot) > 0$ and $\varrho_1(\cdot) \le \varrho_2(\cdot)$ or $\varrho_1(\cdot) \le 0$ and $\varrho_1(\cdot) \le 0 \le \varrho_2(\cdot)$. In any case $\varrho_1(\cdot) \le \varrho_2(\cdot)$. ∎

Proof of Proposition 11. (i)$\Rightarrow$(ii) Since u_1 and u_2 are affine and non-constant, if $\succsim_1$ is intrinsically equivalent to $\succsim_2$, then $u_1 \approx u_2$. W.l.o.g. choose $u_1 = u_2 = u$. For all $\left(x_o, (x_i)_{i\in I}\right) \in \mathcal{X}$ choose c such that $c \sim_2 \left(x_o, (x_i)_{i\in I}\right)$, i.e. $u(c) = u(x_o) + \varrho_2\left(x_o, (x_i)_{i\in I}\right)$.

If $\varrho_2\left(x_o, \mu_{(x_i)_{i\in I}}\right) < 0$,

$$u(x_o) + \varrho_2\left(x_o, \mu_{(x_i)_{i\in I}}\right) < u(x_o)$$
$$x_o \succ_2 c \sim_2 \left(x_o, (x_i)_{i\in I}\right)$$
$$x_o \succ_1 c \succsim_1 \left(x_o, (x_i)_{i\in I}\right)$$
$$u(x_o) + \varrho_1\left(x_o, \mu_{(x_i)_{i\in I}}\right) \le u(c) < u(x_o)$$

then $\varrho_1\left(x_o, \mu_{(x_i)_{i\in I}}\right) \le u(c) - u(x_o) = \varrho_2\left(x_o, \mu_{(x_i)_{i\in I}}\right)$. Analogously, if $\varrho_2\left(x_o, \mu_{(x_i)_{i\in I}}\right) > 0$,

$$u(x_o) + \varrho_2\left(x_o, \mu_{(x_i)_{i\in I}}\right) > u(x_o)$$
$$x_o \prec_2 c \sim_2 \left(x_o, (x_i)_{i\in I}\right)$$
$$x_o \prec_1 c \precsim_1 \left(x_o, (x_i)_{i\in I}\right)$$
$$u(x_o) + \varrho_1\left(x_o, \mu_{(x_i)_{i\in I}}\right) \ge u(c) > u(x_o)$$

then $\varrho_1\left(x_o, \mu_{(x_i)_{i\in I}}\right) \ge u(c) - u(x_o) = \varrho_2\left(x_o, \mu_{(x_i)_{i\in I}}\right)$.

Conclude that: if $\varrho_2(\cdot) < 0$ then $\varrho_1(\cdot) \le \varrho_2(\cdot) < 0$ hence $|\varrho_1(\cdot)| \ge |\varrho_2(\cdot)|$ and $\varrho_1(\cdot)\varrho_2(\cdot) > 0$, obviously if $\varrho_2(\cdot) = 0$ then $|\varrho_1(\cdot)| \ge 0 = |\varrho_2(\cdot)|$ and $\varrho_1(\cdot)\varrho_2(\cdot) = 0$, finally if $\varrho_2(\cdot) > 0$ then $\varrho_1(\cdot) \ge \varrho_2(\cdot) > 0$ hence $|\varrho_1(\cdot)| \ge |\varrho_2(\cdot)|$ and $\varrho_1(\cdot)\varrho_2(\cdot) > 0$.

(ii)$\Rightarrow$(i) Conversely, $u_1 \approx u_2$ clearly implies that $\succsim_1$ is intrinsically equivalent to $\succsim_2$, and w.l.o.g. we can choose $u_1 = u_2 = u$. Assume $x_o \succ_2 y_o \succsim_2 \left(x_o, (x_i)_{i\in I}\right)$,

$$u(x_o) > u(y_o) \ge u(x_o) + \varrho_2\left(x_o, \mu_{(x_i)_{i\in I}}\right)$$
$$\varrho_2\left(x_o, \mu_{(x_i)_{i\in I}}\right) \le u(y_o) - u(x_o) < 0$$

since ϱ_1 and ϱ_2 are cosigned, then $\varrho_1\left(x_o, \mu_{(x_i)_{i\in I}}\right) \le 0$ and the relation between moduli delivers

$$\varrho_1\left(x_o, \mu_{(x_i)_{i\in I}}\right) \le \varrho_2\left(x_o, \mu_{(x_i)_{i\in I}}\right) \le u(y_o) - u(x_o) < 0$$
$$u(x_o) + \varrho_1\left(x_o, \mu_{(x_i)_{i\in I}}\right) \le u(y_o) < u(x_o)$$
$$x_o \succ_1 y_o \succsim_1 \left(x_o, (x_i)_{i\in I}\right).$$

Analogously, if $x_o \prec_2 y_o \precsim_2 \left(x_o, (x_i)_{i \in I}\right)$,

$$u(x_o) < u(y_o) \leq u(x_o) + \varrho_2\left(x_o, \mu_{(x_i)_{i \in I}}\right)$$

$$\varrho_2\left(x_o, \mu_{(x_i)_{i \in I}}\right) \geq u(y_o) - u(x_o) > 0$$

since ϱ_1 and ϱ_2 are cosigned, then $\varrho_1\left(x_o, \mu_{(x_i)_{i \in I}}\right) \geq 0$ and the relation between moduli delivers

$$\varrho_1\left(x_o, \mu_{(x_i)_{i \in I}}\right) \geq \varrho_2\left(x_o, \mu_{(x_i)_{i \in I}}\right) \geq u(y_o) - u(x_o) > 0$$

$$u(x_o) + \varrho_1\left(x_o, \mu_{(x_i)_{i \in I}}\right) \geq u(y_o) > u(x_o)$$

$$x_o \prec_1 y_o \precsim_1 \left(x_o, (x_i)_{i \in I}\right)$$

as wanted. $\blacksquare$

Proof of Theorem 8. We only prove sufficiency, the converse being easy. By Theorem 2 there exist two non-constant affine functions $u, v : C \to \mathbb{R}$, a diago-null function $\varrho : \operatorname{pim}(v(C)) \to \mathbb{R}$ increasing in the first component and decreasing (w.r.t. stochastic dominance) in the second, and $p_0, p_1, ..., p_T \geq 0$ with $\sum_{t=0}^{T} p_t = 1$, such that v represents $\dot{\precsim}$ and the function $W : \mathcal{F} \to \mathbb{R}$, defined by

$$W\left(f_o, (f_i)_{i \in I}\right) = \sum_{t=0}^{T} p_t \left[u(f_o(t)) + \varrho\left(v(f_o(t)), \sum_{i \in I} \delta_{v(f_i(t))}\right)\right]$$

for all $\left(f_o, (f_i)_{i \in I}\right) \in \mathcal{F}$, represents $\precsim$ and satisfies $W(\mathcal{F}) = u(C)$. Essentiality of $\{0\}$ implies $p_0 \in (0, 1)$, and so

$$V\left(f_o, (f_i)_{i \in I}\right) = u(f_o(0)) + \varrho\left(v(f_o(0)), \sum_{i \in I} \delta_{v(f_i(0))}\right) +$$

$$+ \sum_{t=1}^{T} \frac{p_t}{p_0}\left[u(f_o(t)) + \varrho\left(v(f_o(t)), \sum_{i \in I} \delta_{v(f_i(t))}\right)\right] = \frac{W\left(f_o, (f_i)_{i \in I}\right)}{p_0}$$

for all $\left(f_o, (f_i)_{i \in I}\right) \in \mathcal{F}$, represents $\precsim$ and satisfies $V(\mathcal{F}) = p_0^{-1} u(C)$.

Set $b_t = p_t/p_0$ for all $t \in T_0$ and notice that $b_0 = 1$, $p_0 = \left(\sum_{t=0}^{T} b_t\right)^{-1}$, $\sum_{t=0}^{T} b_t > 1$ and

$$V\left(f_o, (f_i)_{i \in I}\right) = \sum_{t=0}^{T} b_t \left[u(f_o(t)) + \varrho\left(v(f_o(t)), \sum_{i \in I} \delta_{v(f_i(t))}\right)\right]$$

for all $\left(f_o, (f_i)_{i \in I}\right) \in \mathcal{F}$, represents $\precsim$ and satisfies $V(\mathcal{F}) = \left(\sum_{t=0}^{T} b_t\right) u(C)$. If $T = 1$, then $b_1 > 0$ and (28) holds with $\beta = b_1$. If $T > 1$, arbitrarily fix $a \in \mathcal{A}_o$, and for all $t < T$ and consider the preferences $\succsim^t$ on $C \times C$ defined by

$$(c, c') \succsim^t (\bar{c}, \bar{c}') \Leftrightarrow \begin{bmatrix} a(\tau) & \text{if } \tau \neq t, t+1 \\ c & \text{if } \tau = t \\ c' & \text{if } \tau = t+1 \end{bmatrix} \succsim \begin{bmatrix} a(\tau) & \text{if } \tau \neq t, t+1 \\ \bar{c} & \text{if } \tau = t \\ \bar{c}' & \text{if } \tau = t+1 \end{bmatrix}.$$

By Axiom 1, all preferences $\succsim^t$ for $t < T$ coincide. Moreover, $\succsim^t$ is represented, for all $t < T$, by the function

$$V\begin{pmatrix} a(\tau) & \text{if } \tau \neq t, t+1 \\ c & \text{if } \tau = t \\ c' & \text{if } \tau = t+1 \end{pmatrix} = \sum_{\tau \neq t, t+1} b_\tau u(a(\tau)) + b_t u(c) + b_{t+1} u(c'), \quad \forall (c, c') \in C \times C$$

and so by the function

$$U^t \left(c, c' \right) = b_t u \left(c \right) + b_{t+1} u \left(c' \right), \quad \forall \left(c, c' \right) \in C \times C.$$

Then $b_0 = 1$ implies $b_t \neq 0$ for all $t < T$. Let, *per contra*, $\tau = \min \{ t < T : b_t = 0 \}$ and $c, c' \in C$ be such that $u \left(c' \right) > u \left(c \right)$. It follows that $0 < \tau < T$ and $b_{\tau - 1} > 0$, thus

$$b_{\tau-1} u \left(c' \right) + b_\tau u \left(c \right) > b_{\tau-1} u \left(c \right) + b_\tau u \left(c \right)$$
$$\left(c', c \right) \succ^{\tau-1} \left(c, c \right)$$
$$\left(c', c \right) \succ^{\tau} \left(c, c \right)$$
$$b_\tau u \left(c' \right) + b_{\tau+1} u \left(c \right) > b_\tau u \left(c \right) + b_{\tau+1} u \left(c \right)$$
$$b_{\tau+1} u \left(c \right) > b_{\tau+1} u \left(c \right)$$

which is absurd. Finally, the uniqueness of the subjective probability in an Anscombe-Aumann representation implies that, for all $\tau, t < T$, $b_t \left(b_t + b_{t+1} \right)^{-1} = b_\tau \left(b_\tau + b_{\tau+1} \right)^{-1}$, $b_t b_\tau + b_t b_{\tau+1} = b_\tau b_t + b_\tau b_{\tau+1}$, and

$$\frac{b_{\tau+1}}{b_\tau} = \frac{b_{t+1}}{b_t} \tag{40}$$

The proof of sufficiency is concluded by setting $\beta = b_1$ and applying (40) $T - 1$ times. ∎

Lemma 12 *A binary relation $\succsim$ on $\mathcal{F}$ satisfies Axioms A.1-A.6 and F.1 if and only if there exist a non-constant affine function $u : C \to \mathbb{R}$, a diago-null function $\varrho : \mathrm{pim} \left(u \left(C \right) \right) \to \mathbb{R}$, and a probability P on Σ, such that the function $V : \mathcal{F} \to \mathbb{R}$, defined by*

$$V \left(f_o, \left(f_i \right)_{i \in I} \right) = \int_S \left[u \left(f_o \left(s \right) \right) + \varrho \left(u \left(f_o \left(s \right) \right), \mu_{\left(u \left(f_i \left(s \right) \right) \right)_{i \in I}} \right) \right] dP \left(s \right) \tag{41}$$

for all $\left(f_o, \left(f_i \right)_{i \in I} \right) \in \mathcal{F}$, represents $\succsim$ and satisfies $V \left(\mathcal{F} \right) = u \left(C \right)$.

The triplet $\left(\hat{u}, \hat{\varrho}, \hat{P} \right)$ is another representation of $\succsim$ in the above sense if and only if $\hat{P} = P$ and there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that $\hat{u} = \alpha u + \beta$, and

$$\hat{\varrho} \left(z, \sum_{i \in I} \delta_{z_i} \right) = \alpha \varrho \left(\alpha^{-1} \left(z - \beta \right), \sum_{i \in I} \delta_{\alpha^{-1} \left(z_i - \beta \right)} \right)$$

for all $\left(z, \sum_{i \in I} \delta_{z_i} \right) \in \mathrm{pim} \left(\hat{u} \left(C \right) \right)$.

Proof. By Lemma 9, there exist a non-constant affine function $u : C \to \mathbb{R}$, a function $r : \mathcal{X} \to \mathbb{R}$ with $r \left(c_{I_o} \right) = 0$ for all $c \in C$ and $I \in \wp \left(N \right)$, and a probability P on Σ, such that the functional $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V \left(f \right) = \int_S \left[u \left(f_o \left(s \right) \right) dP \left(s \right) + r \left(f_o \left(s \right), \left(f_i \left(s \right) \right)_{i \in I} \right) \right] dP \left(s \right)$$

for all $f = \left(f_o, \left(f_i \right)_{i \in I} \right) \in \mathcal{F}$, represents $\succsim$ and satisfies $V \left(\mathcal{F} \right) = u \left(C \right)$.

Next we show that if $\left(x_o, \left(x_i \right)_{i \in I} \right), \left(y_o, \left(y_j \right)_{j \in J} \right) \in \mathcal{X}$, $u \left(x_o \right) = u \left(y_o \right)$, and $\mu_{\left(u \left(x_i \right) \right)_{i \in I}} = \mu_{\left(u \left(y_j \right) \right)_{j \in J}}$, then $r \left(x_o, \left(x_i \right)_{i \in I} \right) = r \left(y_o, \left(y_j \right)_{j \in J} \right)$. Therefore, for $\left(z, \mu \right) \in \mathrm{pim} \left(u \left(C \right) \right)$, it is well posed to define

$$\varrho \left(z, \mu \right) = r \left(x_o, \left(x_i \right)_{i \in I} \right)$$

provided $z = u(x_o)$ and $\mu = \mu_{(u(x_i))_{i \in I}}$. But first notice that at least one $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$ such that $z = u(x_o)$ and $\mu = \mu_{(u(x_i))_{i \in I}}$ exists for every $(z, \mu) \in \operatorname{pim}(u(C))$.[58] If $\mu_{(u(x_i))_{i \in I}} = \mu_{(u(y_j))_{j \in J}} = 0$, then $I = J = \emptyset$. In this case,

$$r\left(x_o, (x_i)_{i \in I}\right) = r(x_o) = 0 = r(y_o) = r\left(y_o, (y_j)_{j \in J}\right)$$

Else if $\mu_{(u(x_i))_{i \in I}} = \mu_{(u(y_j))_{j \in J}} \neq 0$, then, by Lemma 7, there is a bijection $\pi : I \to J$ such that $u(x_i) = u\left(y_{\pi(i)}\right)$ for all $i \in I$. Then $y_{\pi(i)} \sim x_i$ and $y_o \sim x_o$. Axiom F.1 guarantees that

$$\left(x_o, (x_i)_{i \in I}\right) \sim \left(y_o, \left(y_{\pi(i)}\right)_{i \in I}\right).$$

Consider the inverse bijection $\pi^{-1} : J \to I$ and notice that $y_j = y_{\pi(\pi^{-1}(j))}$ for all $j \in J$, then Axiom A.6 delivers $\left(y_o, \left(y_{\pi(i)}\right)_{i \in I}\right) \sim \left(y_o, (y_j)_{j \in J}\right)$ and

$$u(x_o) + r\left(x_o, (x_i)_{i \in I}\right) = u(y_o) + r\left(y_o, (y_j)_{j \in J}\right)$$

which, together with $u(x_o) = u(y_o)$, delivers $r\left(x_o, (x_i)_{i \in I}\right) = r\left(y_o, (y_j)_{j \in J}\right)$. As wanted.

If $z \in u(C)$ and $0 \leq n \leq |N|$, take $c \in C$ such that $u(c) = z$ and $I \in \wp(N)$ such that $|I| = n$, then $\varrho(z, n\delta_z) = r(c_{I_o}) = 0$.

Therefore ϱ is diago-null and $V(f) = \int_S \left[u(f_o(s)) + \varrho\left(u(f_o(s)), \mu_{(u(f_i(s)))_{i \in I}}\right)\right] dP(s)$ for all $f \in \mathcal{F}$. This completes the proof of the sufficiency part of the Theorem.

As for the necessity part, we just have to show that a preference represented by (41) satisfies Axioms A.6 and F.1 (the rest descends from Lemma 9, by setting $r\left(x_o, (x_i)_{i \in I}\right) = \varrho\left(u(x_o), \mu_{(u(x_i))_{i \in I}}\right)$ for all $\left(x_o, (x_i)_{i \in I}\right) \in \mathcal{X}$ and observing that, for all $I \in \wp(N)$ and $c \in C$, $r(c_{I_o}) = \varrho\left(u(c), |I|\delta_{u(c)}\right) = 0$). Let $\left(x_o, (x_i)_{i \in I}\right), \left(y_o, (y_j)_{j \in J}\right) \in \mathcal{X}$ be such that $x_o \sim y_o$ and there is a bijection $\sigma : J \to I$ such that for every $j \in J$, $y_j \sim x_{\sigma(j)}$. If $I = \emptyset$, then $J = \emptyset$ and $x = (x_o) \sim (y_o) = y$. Else

$$\mu_{(u(y_j))_{j \in J}} = \sum_{j \in J} \delta_{u(y_j)} = \sum_{j \in J} \delta_{u\left(x_{\sigma(j)}\right)} = \sum_{i \in I} \delta_{u(x_i)} = \mu_{(u(x_i))_{i \in I}}$$

since also $u(x_o) = u(y_o)$,

$$\varrho\left(u(x_o), \mu_{(u(x_i))_{i \in I}}\right) = \varrho\left(u(y_o), \mu_{(u(y_j))_{j \in J}}\right)$$
$$u(x_o) + \varrho\left(u(x_o), \mu_{(u(x_i))_{i \in I}}\right) = u(y_o) + \varrho\left(u(y_o), \mu_{(u(y_j))_{j \in J}}\right)$$
$$\left(x_o, (x_i)_{i \in I}\right) \sim \left(y_o, (y_j)_{j \in J}\right).$$

From the special case in which $x_o = y_o$ and $y_j = x_{\sigma(j)}$ for all $j \in J$, it follows that Axiom A.6 holds. From the special case in which $I = J$ and σ is the identity, it follows that Axiom F.1 holds.

The proof of the uniqueness part is very similar to that of Proposition 1. ∎

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[58] In any case take $x_o \in u^{-1}(z)$. If $\mu = 0$ take $I = \emptyset$. Else if $\mu = \sum_{k=1}^n \delta_{z_k}$ for some $n \geq 1$, take a subset $I = \{i_1, ..., i_n\}$ of N with cardinality n and arbitrarily choose $x_{i_k} \in u^{-1}(z_k)$ for all $k = 1, ..., n$.

Proof of Theorem 9. By Lemma 12 there exist a non-constant affine function $u : C \to \mathbb{R}$, a diago-null function $\varrho : \text{pim}(u(C)) \to \mathbb{R}$, and a probability P on Σ, such that the function $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V\left(f_o, (f_i)_{i \in I}\right) = \int_S \left[u(f_o(s)) + \varrho\left(u(f_o(s)), \mu_{(u(f_i(s)))_{i \in I}}\right)\right] dP(s)$$

for all $\left(f_o, (f_i)_{i \in I}\right) \in \mathcal{F}$, represents $\succsim$ and satisfies $V(\mathcal{F}) = u(C)$.

For all $(z, \mu, \nu) \in \text{pid}(u(C))$ set $\xi(z, \mu, \nu) = \varrho(z, \mu + \nu)$, clearly ξ is well defined and $\xi(z, 0, n\delta_z) = \varrho(z, n\delta_z) = 0$ for all $z \in u(C)$ and $0 \le n \le |N|$, that is ξ is diago-null. Next we show that ξ is increasing in the second component w.r.t. stochastic dominance.

Let $\left(z, \sum_{i \in I} \delta_{a_i}, \sum_{l \in L} \delta_{z_l}\right), \left(z, \sum_{j \in J} \delta_{b_j}, \sum_{l \in L} \delta_{z_l}\right) \in \text{pid}(u(C))$ and assume $\sum_{i \in I} \delta_{a_i}$ stochastically dominates $\sum_{j \in J} \delta_{b_j}$.[59]

If $I = J = \emptyset$, then $\xi\left(z, \sum_{i \in I} \delta_{a_i}, \sum_{l \in L} \delta_{z_l}\right) = \xi\left(z, 0, \sum_{l \in L} \delta_{z_l}\right) = \xi\left(z, \sum_{j \in J} \delta_{b_j}, \sum_{l \in L} \delta_{z_l}\right)$.

Else if $I, J \ne \emptyset$, then $|I| = |J|$, and w.l.o.g. we can assume $I = \{i_1, ..., i_n\}$ and $J = \{j_1, ..., j_n\}$ with $a_{i_1} \le ... \le a_{i_n} < z$ and $b_{j_1} \le ... \le b_{j_n} < z$, and $F_{\mathbf{a}} \le F_{\mathbf{b}}$, by Lemma 5, $a_{i_k} \ge b_{j_k}$ for all $k = 1, ..., n$. Let $\left(x_o, (y_{j_k})_{k=1}^n, (w_l)_{l \in L}\right), \left(x_o, (x_{j_k})_{k=1}^n, (w_l)_{l \in L}\right) \in \mathcal{X}$, be such that $u(x_o) = z$, $u(y_{j_k}) = b_{j_k}$ for all $k = 1, ..., n$, $u(w_l) = z_l$ for all $l \in L$, $u(x_{j_k}) = a_{i_k}$ for all $k = 1, ..., n$. Then $x_o \succ x_{j_k} \succsim y_{j_k}$ for $k = 1, ..., n$, and n applications of Axiom F.2 deliver

$$\left(x_o, x_{j_1}, x_{j_2}, ..., x_{j_n}, (w_l)_{l \in L}\right) \succsim \left(x_o, y_{j_1}, x_{j_2}, ..., x_{j_n}, (w_l)_{l \in L}\right) \succsim ... \succsim \left(x_o, y_{j_1}, ..., y_{j_n}, (w_l)_{l \in L}\right)$$

that is $\left(x_o, (x_{j_k})_{k=1}^n, (w_l)_{l \in L}\right) \succsim \left(x_o, (y_{j_k})_{k=1}^n, (w_l)_{l \in L}\right)$ hence

$$u(x_o) + \varrho\left(u(x_o), \sum_{k=1}^n \delta_{u(x_{j_k})} + \sum_{l \in L} \delta_{u(w_l)}\right) \ge u(x_o) + \varrho\left(u(x_o), \sum_{k=1}^n \delta_{u(y_{j_k})} + \sum_{l \in L} \delta_{u(w_l)}\right)$$

and

$$\xi\left(z, \sum_{i \in I} \delta_{a_i}, \sum_{l \in L} \delta_{z_l}\right) = \varrho\left(z, \sum_{k=1}^n \delta_{a_{i_k}} + \sum_{l \in L} \delta_{z_l}\right) = \varrho\left(u(x_o), \sum_{k=1}^n \delta_{u(x_{j_k})} + \sum_{l \in L} \delta_{u(w_l)}\right)$$

$$\ge \varrho\left(u(x_o), \sum_{k=1}^n \delta_{u(y_{j_k})} + \sum_{l \in L} \delta_{u(w_l)}\right) = \xi\left(z, \sum_{j \in J} \delta_{b_j}, \sum_{l \in L} \delta_{z_l}\right).$$

A similar argument shows that Axiom F.2 also delivers decreasing monotonicity of ξ in the third component w.r.t. stochastic dominance.

This completes the proof of the sufficiency part.

As for the necessity part, notice that for all $(z, \mu) \in \text{pim}(u(C))$, $\sum_{r \in \text{supp}(\mu):r<z} \mu(r) \delta_r$ and $\sum_{r \in \text{supp}(\mu):r \ge z} \mu(r) \delta_r$ are positive integer measures finitely supported in $u(C) \cap (-\infty, z)$ and $u(C) \cap [z, \infty)$ respectively, and their sum μ has total mass bounded by $|N|$, that is $\left(z, \sum_{r \in \text{supp}(\mu):r<z} \mu(r) \delta_r, \sum_{r \in \text{supp}(\mu):r \ge z} \mu(r) \delta_r\right) \in \text{pid}(u(C))$. Define

$$\varrho(z, \mu) \equiv \xi\left(z, \sum_{r \in \text{supp}(\mu):r<z} \mu(r) \delta_r, \sum_{r \in \text{supp}(\mu):r \ge z} \mu(r) \delta_r\right)$$

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[59]Notice that since $\left(z, \sum_{i \in I} \delta_{a_i}, \sum_{l \in L} \delta_{z_l}\right)$ and $\left(z, \sum_{j \in J} \delta_{b_j}, \sum_{l \in L} \delta_{z_l}\right)$ belong to $\text{pid}(u(C))$ we can assume that I, J, L are finite subsets of N with $I \cap L = \emptyset$ and $J \cap L = \emptyset$.

and notice that $\varrho\left(z, n\delta_z\right) = \xi\left(z, 0, n\delta_z\right) = 0$ for all z in $u\left(C\right)$ and all non-negative integers $n \le |N|$. Thus $u : C \to \mathbb{R}$ is a non-constant affine function, a $\varrho : \mathrm{pim}\left(u\left(C\right)\right) \to \mathbb{R}$ is a diago-null function, and P is a probability on Σ, such that the function $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V\left(f_o, \left(f_i\right)_{i \in I}\right) = \int_S \left[u\left(f_o\left(s\right)\right) + \varrho\left(u\left(f_o\left(s\right)\right), \sum_{i \in I} \delta_{u\left(f_i\left(s\right)\right)}\right)\right] dP\left(s\right)$$

$$= \int_S \left[u\left(f_o\left(s\right)\right) + \xi\left(u\left(f_o\left(s\right)\right), \sum_{i \in I : u\left(f_i\left(s\right)\right) < u\left(f_o\left(s\right)\right)} \delta_{u\left(f_i\left(s\right)\right)}, \sum_{i \in I : u\left(f_i\left(s\right)\right) \ge u\left(f_o\left(s\right)\right)} \delta_{u\left(f_i\left(s\right)\right)}\right)\right] dP\left(s\right)$$

for all $\left(f_o, \left(f_i\right)_{i \in I}\right) \in \mathcal{F}$, represents $\succsim$ and satisfies $V\left(\mathcal{F}\right) = u\left(C\right)$. Lemma 12 guarantees that $\succsim$ satisfies Axioms A.1-A.6 and F.1.

Next we show that $\succsim$ satisfies Axiom F.2. Let $\left(x_o, \left(x_i\right)_{i \in I}\right) \in \mathcal{X}$, $j \in I$, and $c \in C$.

If $c \succsim x_j \succsim x_o$. Then $u\left(c\right) \ge u\left(x_j\right) \ge u\left(x_o\right)$, and, by Lemma 7, $\sum_{i \in I - \{j\} : u\left(x_i\right) \ge u\left(x_o\right)} \delta_{u\left(x_i\right)} + \delta_{u\left(c\right)}$ stochastically dominates $\sum_{i \in I : u\left(x_i\right) \ge u\left(x_o\right)} \delta_{u\left(x_i\right)}$. Therefore

$$\xi\left(u\left(x_o\right), \sum_{i \in I : u\left(x_i\right) < u\left(x_o\right)} \delta_{u\left(x_i\right)}, \sum_{i \in I : u\left(x_i\right) \ge u\left(x_o\right)} \delta_{u\left(x_i\right)}\right) \ge$$

$$\ge \xi\left(u\left(x_o\right), \sum_{i \in I : u\left(x_i\right) < u\left(x_o\right)} \delta_{u\left(x_i\right)}, \sum_{i \in I - \{j\} : u\left(x_i\right) \ge u\left(x_o\right)} \delta_{u\left(x_i\right)} + \delta_{u\left(c\right)}\right)$$

and $\left(x_o, \left(x_i\right)_{i \in I}\right) \succsim \left(x_o, \left(x_i\right)_{i \in I \setminus \{j\}}, c_{\{j\}}\right)$. Else, if $c \precsim x_j \prec x_o$, then $u\left(c\right) \le u\left(x_j\right) < u\left(x_o\right)$, and, by Lemma 7, $\sum_{i \in I : u\left(x_i\right) < u\left(x_o\right)} \delta_{u\left(x_i\right)}$ stochastically dominates $\sum_{i \in I - \{j\} : u\left(x_i\right) < u\left(x_o\right)} \delta_{u\left(x_i\right)} + \delta_{u\left(c\right)}$. Therefore

$$\xi\left(u\left(x_o\right), \sum_{i \in I : u\left(x_i\right) < u\left(x_o\right)} \delta_{u\left(x_i\right)}, \sum_{i \in I : u\left(x_i\right) \ge u\left(x_o\right)} \delta_{u\left(x_i\right)}\right) \ge$$

$$\ge \xi\left(u\left(x_o\right), \sum_{i \in I - \{j\} : u\left(x_i\right) < u\left(x_o\right)} \delta_{u\left(x_i\right)} + \delta_{u\left(c\right)}, \sum_{i \in I : u\left(x_i\right) \ge u\left(x_o\right)} \delta_{u\left(x_i\right)}\right)$$

and $\left(x_o, \left(x_i\right)_{i \in I}\right) \succsim \left(x_o, \left(x_i\right)_{i \in I \setminus \{j\}}, c_{\{j\}}\right)$.

This completes the proof. ∎

The proof of Proposition 13 is very similar to the one of Proposition 1, thus omitted.

11.5 Social Economics: Proofs and Related Analysis

For each agent $i \in I$, set

$$W_i\left(x, y, z\right) \equiv \sum_{s \in S_0} \pi_s u_i\left(x_s, y_s\right) + \sum_{s \in S_0} \pi_s \left[\gamma_i\left(v\left(x_s\right) - z_s\right)\right] \tag{42}$$

for all $\left(x, y, z\right) \in \mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1} \times \mathbb{R}^{S+1}$, where $\pi_0 = 1$ and $\pi_s = \beta p_s$ for all $s \in S$. Notice that

$$V_i\left(c, e\right) = \sum_{s \in S_0} \pi_s u_i\left(c_{i,s}, e_{i,s}\right) + \sum_{s \in S_0} \pi_s \gamma_i\left(v\left(c_{i,s}\right) - \int_I \left(v \circ c_s\right) d\lambda\right)$$

$$= W_i\left(c_i, e_i, \left[\int_I \left(v \circ c_s\right) d\lambda\right]_{s \in S_0}\right)$$

for all $\left(c, e\right) \in L_+^{S+1} \times L_+^{S+1}$.

Lemma 13 *If (c^*, e^*) is a social equilibrium, then $(c^*, e^*) \in L^{S+1} \times L^{S+1}$ and (c_i^*, e_i^*) is a solution of problem*

$$\max_{(x,y) \in B_i} W_i(x, y, z)$$

where $z_s \equiv \int_I (v \circ c_s^*) \, d\lambda$ for all $s \in S_0$, for λ-almost all $i \in I$. The converse is true up to a λ-negligible variation of (c^*, e^*).[60]

The simple proof is omitted.

Lemma 14 *If H.1 holds, then B_i is compact and nonempty for all $i \in I$.*

Proof. Since $F_{i,s}$ is increasing, then

$$B_i = \left\{ (x, y) \in \mathbb{R}_+^{S+1} \times \prod_{s=0}^S [0, h_{i,s}] : F_{i,0}(y_0) \geq x_0, \ x_s = F_{i,s}(y_s) + R(F_{i,0}(y_0) - x_0) \ \forall s \in S \right\}$$

$$= \left([0, F_{i,0}(h_{i,0})] \times \prod_{s=1}^S [0, F_{i,s}(h_{i,s}) + RF_{i,0}(h_{i,0})] \times \prod_{s=0}^S [0, h_{i,s}] \right)$$

$$\cap \left\{ (x, y) \in \mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1} : F_{i,0}(y_0) - x_0 \geq 0 \right\}$$

$$\cap \left\{ (x, y) \in \mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1} : F_{i,1}(y_1) + R(F_{i,0}(y_0) - x_0) - x_1 = 0 \right\}$$

$$\dots$$

$$\cap \left\{ (x, y) \in \mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1} : F_{i,S}(y_S) + R(F_{i,0}(y_0) - x_0) - x_S = 0 \right\}$$

which is compact since the functions

$$(x, y) \mapsto F_{i,0}(y_0) - x_0$$
$$(x, y) \mapsto F_{i,s}(y_s) + R(F_{i,0}(y_0) - x_0) - x_s$$

are continuous for all $s \in S$. Moreover, for all $y \in \prod_{s=0}^S [0, h_{i,s}]$, $\left((F_{i,s}(y_s))_{s=0}^S, (y_s)_{s=0}^S \right) \in B_i$, which implies $B_i \neq \emptyset$. ∎

Proof of Theorem 6. For each agent i, consider the strategy set $B_i \subseteq \mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1}$ and the payoff function

$$W_i(x, y, z) = \sum_{s \in S_0} \pi_s u_i(x_s, y_s) + \sum_{s \in S_0} \pi_s [\gamma_i(v(x_s) - z_s)]$$

for all $(x, y, z) \in \mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1} \times \mathbb{R}^{S+1}$, where $\pi_0 = 1$ and $\pi_s = \beta p_s$ for all $s \in S$.

Since $\mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1}$ and $\mathbb{R}^{S+1}$ are Polish spaces, assumptions 2.1 and 2.2 of Balder (1995) hold.

Since B_i is nonempty and compact for all $i \in I$, assumption 2.3 of Balder (1995) holds.

For every $i \in I$, $W_i : \mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1} \times \mathbb{R}^{S+1} \to \mathbb{R}$ is continuous, therefore assumptions 2.4 and 2.6 of Balder (1995) hold.

Assumptions H.1.ii and H.2.iii guarantee that the real valued functions on $I \times \left(\mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1} \right)$ defined by

$$(i, (x, y)) \mapsto F_{i,0}(y_0) - x_0$$
$$(i, (x, y)) \mapsto F_{i,s}(y_s) + R(F_{i,0}(y_0) - x_0) - x_s$$

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[60]A λ-negligible variation of a function on a measure space (I, Λ, λ) is a function that coincides λ-almost everywhere with the original one.

are Caratheodory functions (see [6, p. 153]) for all $s \in S$, hence they are $\Lambda \times \mathcal{B}\left(\mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1}\right)$-measurable and the graphs of the correspondences

$$i \mapsto \left\{(x,y) \in \mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1} : F_{i,0}(y_0) - x_0 \geq 0\right\}$$
$$i \mapsto \left\{(x,y) \in \mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1} : F_{i,s}(y_s) + R\left(F_{i,0}(y_0) - x_0\right) - x_s = 0\right\}$$

are $\Lambda \times \mathcal{B}\left(\mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1}\right)$-measurable.

Moreover, the functions $i \mapsto F_s(i, h_s(i))$ are Λ-measurable for all $s \in S_0$, since $i \mapsto (i, h_s(i))$ is Λ-measurable on I and $(i,t) \mapsto F_{i,s}(t)$ is a Caratheodory function on $I \times \mathbb{R}_+$. Therefore the graph of the correspondence

$$i \mapsto [0, F_{i,0}(h_{i,0})] \times \prod_{s=1}^{S} [0, F_{i,s}(h_{i,s}) + R F_{i,0}(h_{i,0})] \times \prod_{s=0}^{S} [0, h_{i,s}]$$

is $\Lambda \times \mathcal{B}\left(\mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1}\right)$-measurable.

Since

$$B_i = \left\{(x,y) \in \mathbb{R}_+^{S+1} \times \prod_{s=0}^{S} [0, h_{i,s}] : F_{i,0}(y_0) \geq x_0, \ x_s = F_{i,s}(y_s) + R\left(F_{i,0}(y_0) - x_0\right) \ \forall s \in S\right\}$$

$$= \left([0, F_{i,0}(h_{i,0})] \times \prod_{s=1}^{S} [0, F_{i,s}(h_{i,s}) + R F_{i,0}(h_{i,0})] \times \prod_{s=0}^{S} [0, h_{i,s}]\right) \cap$$
$$\cap \left\{(x,y) \in \mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1} : F_{i,0}(y_0) - x_0 \geq 0\right\}$$
$$\cap \left\{(x,y) \in \mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1} : F_{i,1}(y_1) + R\left(F_{i,0}(y_0) - x_0\right) - x_1 = 0\right\}$$
$$...$$
$$\cap \left\{(x,y) \in \mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1} : F_{i,S}(y_S) + R\left(F_{i,0}(y_0) - x_0\right) - x_S = 0\right\}$$

for all $i \in I$, then the graph of the correspondence $B : i \mapsto B_i$ is $\Lambda \times \mathcal{B}\left(\mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1}\right)$-measurable and assumption 2.5 of Balder (1995) holds.

For every $z \in \mathbb{R}^{S+1}$, $W_{(\cdot)}(\cdot, z) : (i, (x,y)) \mapsto W_i((x,y), z)$ is a Caratheodory function on $I \times \left(\mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1}\right)$, hence it is $\Lambda \times \mathcal{B}(\mathbb{R}^n)$-measurable, therefore assumption 2.7 of Balder (1995) holds.

Now define $g_s : I \times \mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1} \to \mathbb{R}$ by $g_s(i, (x,y)) = v(x_s)$ for all $s \in S_0$. Let $s \in S_0$, clearly $g_s(i, (\cdot, \cdot))$ is continuous on $\mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1}$ for all $i \in I$ and $g_s(\cdot, (x,y))$ is constant – hence Λ-measurable – on I for all $(x,y) \in \mathbb{R}_+^{S+1} \times \mathbb{R}_+^{S+1}$. Therefore g_s is a Caratheodory function for all $s \in S_0$, in particular, it is $\Lambda \times \mathcal{B}(\mathbb{R}^n)$-measurable. For all $i \in I$

$$\inf_{(x,y) \in B_i} g_s(i, (x,y)) = \inf_{(x,y) \in B_i} v(x_s) \geq v(0)$$

$$\sup_{(x,y) \in B_i} g_s(i, (x,y)) = \sup_{(x,y) \in B_i} v(x_s) \leq \sup_{x_s \in [0, F_{i,s}(h_{i,s}) + RF_{i,0}(h_{i,0})]} v(x_s) = v\left(F_{i,s}(h_{i,s}) + R F_{i,0}(h_{i,0})\right)$$

and $i \mapsto v\left(F_{i,s}(h_{i,s}) + R F_{i,0}(h_{i,0})\right)$ is Λ-measurable and bounded (by H.2.i). Therefore assumption 3.4.2 of Balder (1995) holds.

Finally, nonatomicity of λ guarantees that assumption 3.4.1 of Balder (1995) holds too.

Therefore, by Theorem 3.4.1 of Balder (1995) there exists a λ-measurable a.e. selection (c^*, e^*) of the correspondence

$$B : i \mapsto B_i$$

such that for λ-almost all i,

$$(c_i^*, e_i^*) \in \arg\max_{(x,y) \in B_i} \left(\sum_{s \in S_0} \pi_s u_i(x_s, y_s) + \sum_{s \in S_0} \pi_s \left[\gamma_i \left(v(x_s) - m_s(c^*, e^*) \right) \right] \right)$$

where

$$m_s(c^*, e^*) = \int_I g_s(\iota, (c_\iota^*, e_\iota^*)) \, d\lambda(\iota) = \int_I v(c_{\iota,s}^*) \, d\lambda(\iota).$$

Since B_i is never empty, wlog, we can assume that $(c_i^*, e_i^*) \in B_i$ for all $i \in I$. Then, by Corollary, we only have to check that (c^*, e^*) is bounded. This is easily obtained, setting

$$\Xi = \max_{s \in S_0} \left(\sup_{i \in I} (F_{i,s}(h_{i,s}) + h_{i,s}) \right)$$

which is finite by H.2.i, and observing that, for all $i \in I$,

$$B_i \subseteq [0, F_{i,0}(h_{i,0})] \times \prod_{s=1}^{S} [0, F_{i,s}(h_{i,s}) + R F_{i,0}(h_{i,0})] \times \prod_{s=0}^{S} [0, h_{i,s}]$$

$$\subseteq [0, \Xi] \times \prod_{s=1}^{S} [0, (1+R)\Xi] \times \prod_{s=0}^{S} [0, \Xi].$$

$\blacksquare$

Proof of Lemma 3. For all $i \in I$, problem (21) is equivalent to

$$\max_{0 \le y \le h_i} u_i(F_i(y), y).$$

Setting $U_i(y) = u_i(F_i(y), y)$ for all $y \in \mathbb{R}_+$ it is easily checked that

$$U_i'(y) = \nabla u_i(F_i(y), y) \begin{bmatrix} F_i'(y) \\ 1 \end{bmatrix} = F_i'(y) \frac{\partial u_i}{\partial x}(F_i(y), y) + \frac{\partial u_i}{\partial y}(F_i(y), y)$$

and

$$U_i''(y) = \begin{bmatrix} F_i'(y) \\ 1 \end{bmatrix}^{\mathsf{T}} \nabla^2 u_i(F_i(y), y) \begin{bmatrix} F_i'(y) \\ 1 \end{bmatrix} + F_i''(y) \frac{\partial u_i}{\partial x}(F_i(y), y).$$

H.3 guarantees that $U_i'' < 0$ on $(0, h_i)$, in particular, U_i is concave on $[0, h_i]$ and U_i' is strictly decreasing on $(0, h_i)$.

If U_i' never vanishes, since derivatives have the Darboux property, then U_i is either strictly increasing or decreasing on $[0, h_i]$ and the maximum is achieved at $y_i^* = h_i$ or $y_i^* = 0$ (and nowhere else).

If U_i' vanishes at some y_i^* in $(0, h_i)$, then y_i^* is the unique maximum (U_i' is strictly decreasing on $(0, h_i)$).

We can conclude that if an equilibrium profile (c^*, e^*) exists, then it is, λ-a.e. unique since it must satisfy

$$c_i^* = F_i(y_i^*) \text{ and } e_i^* = y_i^*$$

for λ-almost all $i \in I$.

$\blacksquare$

For any function f on $\mathbb{R}$, set

$$D^{\pm} f(t) = \limsup_{h \to 0\pm} \frac{f(t+h) - f(t)}{h} \text{ and } D_{\pm} f(t) = \liminf_{h \to 0\pm} \frac{f(t+h) - f(t)}{h}. \tag{43}$$

Next proposition shows that H.1, H.2, and H.4(iii) imply that $W : E \to L$ is well defined.

Proposition 16 *If H1-H2 hold, then* $W_{(\cdot)}(e) : i \mapsto W_i(e)$ *is Λ-measurable for all $e \in E$. Moreover, $W(e) \in L$ for all $e \in E$ provided*

$$\sup_{i \in I} \left(\sup_{(x,y,z) \in [0,n]^2 \times [-n,n]} |u_i(x,y) + \gamma_i(z)| \right) < \infty \quad \forall n \in \mathbb{N}. \tag{44}$$

Proof. Fix $e \in E$. The function $i \mapsto F_i(e(i))$ is Λ-measurable, since $i \mapsto (i, e(i))$ is Λ-measurable from I to $I \times \mathbb{R}_+$ and $(i,t) \mapsto F_i(t)$ is a Caratheodory function from $I \times \mathbb{R}_+$ to $\mathbb{R}_+$. H.2(i) implies that the function $i \mapsto F_i(e(i))$ is also bounded.

Set $m^* = \int v(F_\iota(e_\iota)) d\lambda(\iota)$.

For every $i \in I$, the real valued function on $\mathbb{R}_+^2$ defined by

$$(x, y) \mapsto u_i(x, y) + \gamma_i(v(x) - m^*)$$

is continuous, and for every $(x, y) \in \mathbb{R}_+^2$, the real valued function on I defined by

$$i \mapsto u_i(x, y) + \gamma_i(v(x) - m^*)$$

is Λ-measurable, hence the real valued function on $I \times \mathbb{R}_+^2$ defined by

$$(i, (x, y)) \mapsto u_i(x, y) + \gamma_i(v(x) - m^*) \tag{45}$$

is $\Lambda \times \mathcal{B}(\mathbb{R}_+^2)$-measurable (being a Caratheodory function).

Conclude that

$$i \mapsto (i, F_i(e_i), e_i)$$

is Λ-measurable from I to $I \times \mathbb{R}_+^2$ (since $i \mapsto F_i(e(i))$ is Λ-measurable), and its composition with (45) delivers measurability of $W_{(\cdot)}(e)$.

Finally

$$\sup_{i \in I} |W_i(e)| = \sup_{i \in I} |u_i(F_i(e_i), e_i) + \gamma_i(v(F_i(e_i)) - m^*)|.$$

By H.2.i, $\Xi = \sup_{i \in I}(F_i(h_i) + h_i) < \infty$ hence $(0, 0) \leq (F_i(e_i), e_i) \leq (F_i(h_i), h_i) \leq (\Xi, \Xi)$ for all $i \in I$, moreover $v(F_i(e_i)) \in [v(0), v(\Xi)]$ and $v(F_i(e_i)) - m^* \in [v(0) - m^*, v(\Xi) - m^*]$ for all $i \in I$, thus

$$\sup_{i \in I} |W_i(e)| \leq \sup_{i \in I} \left(\sup_{(x,y,z) \in [0,\Xi]^2 \times [v(0) - m^*, v(\Xi) - m^*]} |u_i(x, y) + \gamma_i(z)| \right)$$

which is finite if (44) holds. ∎

Lemma 15 *If H.1-H.3 and H.4.i hold, then all social equilibria (c^*, e^*) are such that $c_i^* \geq \hat{c}_i$ and $e_i^* \geq \hat{e}_i$ λ-a.e. If (c^*, e^*) is internal and $D^+\gamma_i > 0$, then, $e_i^* > \hat{e}_i$ and $c_i^* > \hat{c}_i$ λ-a.e..*

Proof. Notice that, by Lemma 13, if a pair $(c^*, e^*) \in L \times L$ is a social equilibrium, then, setting $m^* = \int (v \circ c^*) d\lambda$, (c_i^*, e_i^*) is a solution of problem

$$\max_{(x,y) \in B_i} u_i(x, y) + \gamma_i(v(x) - m^*). \tag{46}$$

For all $i \in I$, problem (46) is equivalent to

$$\max_{0 \leq y \leq h_i} u_i(F_i(y), y) + \gamma_i(v(F_i(y)) - m^*).$$

If (c^*, e^*) is a social equilibrium and $(\hat{c}, \hat{e})$ is the asocial equilibrium, then for λ-almost all $i \in I$,

$$e_i^* \in \arg\max_{0 \leq y \leq h_i} u_i\left(F_i\left(y\right), y\right) + \gamma_i\left(v(F_i\left(y\right)) - m^*\right),$$

$$c_i^* = F\left(e_i^*\right),$$

$$m^* = \int v\left(F_\iota\left(e_\iota^*\right)\right) d\lambda\left(\iota\right),$$

$$\hat{e}_i \in \arg\max_{0 \leq y \leq h_i} u_i\left(F_i\left(y\right), y\right),$$

$$\hat{c}_i = F\left(\hat{e}_i\right).$$

If $e_i^* = h_i$ or $\hat{e}_i = 0$, then $e_i^* \geq \hat{e}_i$ and $c_i^* = F_i\left(e_i^*\right) \geq F_i\left(\hat{e}_i\right) = \hat{c}_i$.

If $e_i^* = 0$ and $\hat{e}_i > 0$, then:

- either U_i' never vanishes in $(0, h_i)$,[61] then U_i is strictly increasing (it cannot be strictly decreasing, otherwise $\hat{e}_i = 0$), but also $\gamma_i\left(v(F_i\left(\cdot\right)) - m^*\right)$ is increasing and the inclusion

$$0 \in \arg\max_{0 \leq y \leq h_i} u_i\left(F_i\left(y\right), y\right) + \gamma_i\left(v(F_i\left(y\right)) - m^*\right) = \arg\max_{0 \leq y \leq h_i} U_i\left(y\right) + \gamma_i\left(v(F_i\left(y\right)) - m^*\right)$$

is absurd,

- or U_i' vanishes at $\hat{e}_i \in (0, h_i)$, then U_i' – being strictly decreasing – must be positive in a right neighborhood of 0, again $u_i\left(F_i\left(\cdot\right), \cdot\right) + \gamma_i\left(v(F_i\left(\cdot\right)) - m^*\right)$ is strictly increasing in a right neighborhood of 0, which is absurd.

It follows that, if $e_i^* = 0$, then $\hat{e}_i = 0$, thus $e_i^* \geq \hat{e}_i$ and $c_i^* = F_i\left(e_i^*\right) \geq F_i\left(\hat{e}_i\right) = \hat{c}_i$.

Finally, if $e_i^* \in (0, h_i)$, then

$$D^+\left[u_i\left(F_i\left(\cdot\right), \cdot\right) + \gamma_i\left(v(F_i\left(\cdot\right)) - m^*\right)\right]\left(e_i^*\right) \leq 0 \tag{47}$$

that is

$$U_i'\left(e_i^*\right) + F_i'\left(e_i^*\right) v'\left(F_i\left(e_i^*\right)\right) D^+\gamma_i(v(F_i\left(e_i^*\right)) - m^*) \leq 0$$

and

$$U_i'\left(e_i^*\right) \leq -F_i'\left(e_i^*\right) v'\left(F_i\left(e_i^*\right)\right) D^+\gamma_i(v(F_i\left(e_i^*\right)) - m^*). \tag{48}$$

By monotonicity, $D^+\gamma_i \geq 0$, therefore $U_i'\left(e_i^*\right) \leq 0$, which implies $e_i^* \geq \hat{e}_i$ because from the proof of Lemma 3 we know that U_i is concave on $[0, h_i]$ with a unique maximum. Again $c_i^* = F_i\left(e_i^*\right) \geq F_i\left(\hat{e}_i\right) = \hat{c}_i$.

Suppose that (c^*, e^*) is an internal social equilibrium and $D^+\gamma_i > 0$. Then (48) delivers $U_i'\left(e_i^*\right) < 0$, which implies $e_i^* > \hat{e}_i$ because from the proof of Lemma 3 we know that U_i is concave on $[0, h_i]$ with a unique maximum. It follows that $c_i^* = F_i\left(e_i^*\right) > F_i\left(\hat{e}_i\right) = \hat{c}_i$. ∎

Proposition 17 *If assumptions H.1-H.4 hold. Then all internal social equilibrium profiles (e^*, c^*) are strongly inefficient.*

Proof. Suppose, *per contra*, that $(c^*, e^*) \in L \times L$ is a social equilibrium with $e^* \in \text{int}\left(E\right)$ and (c^*, e^*) is not strongly inefficient. Let $f : L \to \mathbb{R}$ be defined by $f\left(\xi\right) = \text{essinf}_\lambda\left[\xi - W\left(e^*\right)\right]$ for all $\xi \in L$. Then, $f\left(W\left(e^*\right)\right) = 0$ and $f\left(W\left(e\right)\right) \leq 0$ for all $e \in E$. Moreover, f is a concave niveloid.[62]

Next we show that there is no concave niveloid $f : L \to \mathbb{R}$ such that e^* solves the problem $\max_{e \in E} (f \circ W)(e)$, which is absurd.

First observe that Gateaux differentiability of W guarantees that for all $e \in \mathrm{int}E$ there exists a linear and continuous operator $\nabla W(e) : L \to L$ such that

$$\lim_{t \to 0} \frac{W(e+tk) - W(e)}{t} = \nabla W(e)(k) \in L \tag{49}$$

for all $k \in L$. Arbitrarily choose $e \in \mathrm{int}E$ and $k \in L$, (49) means that

$$\lim_{t \to 0} \left\| \frac{W(e+tk) - W(e)}{t} - \nabla W(e)(k) \right\|_{\sup} = 0.$$

A fortiori, for all $i \in I$,

$$\frac{W_i(e+tk) - W_i(e)}{t} \to \nabla W(e)(k)_i \ \text{ as } t \to 0,$$

but for all $i \in I$

$$\frac{W_i(e+tk) - W_i(e)}{t} = \frac{u_i(F_i(e_i+tk_i), e_i+tk_i) + \gamma_i \left(v(F_i(e_i+tk_i)) - \int v(F_\iota(e_\iota+tk_\iota)) \, d\lambda(\iota) \right)}{t} +$$
$$- \frac{u_i(F_i(e_i), e_i) + \gamma_i \left(v(F_i(e_i)) - \int v(F_\iota(e_\iota)) \, d\lambda(\iota) \right)}{t}.$$

As we show below, this implies that

$$\nabla W(e)(k)_i = k_i U_i'(e_i) + \gamma_i' \left(v(F_i(e_i)) - \int v(F_\iota(e_\iota)) \, d\lambda(\iota) \right) \times \tag{50}$$
$$\times \left(v'(F_i(e_i)) F_i'(e_i) k_i - \int v'(F_\iota(e_\iota)) F_\iota'(e_\iota) k_\iota \, d\lambda(\iota) \right)$$

for λ-almost all $i \in I$.

Rearrange to obtain that for all $i \in I$

$$\frac{W_i(e+tk) - W_i(e)}{t} = \frac{u_i(F_i(e_i+tk_i), e_i+tk_i) - u_i(F_i(e_i), e_i)}{t} +$$
$$+ \frac{\gamma_i \left(v(F_i(e_i+tk_i)) - \int v(F_\iota(e_\iota+tk_\iota)) \, d\lambda(\iota) \right) - \gamma_i \left(v(F_i(e_i)) - \int v(F_\iota(e_\iota)) \, d\lambda(\iota) \right)}{t}.$$

Consider the first term: if $k_i \neq 0$,

$$\frac{u_i(F_i(e_i+tk_i), e_i+tk_i) - u_i(F_i(e_i), e_i)}{t} = k_i \left(\frac{u_i(F_i(e_i+tk_i), e_i+tk_i) - u_i(F_i(e_i), e_i)}{k_i t} \right)$$
$$\to k_i U_i'(e_i) \ \text{ as } t \to 0$$

and the same is true if $k_i = 0$.

For the second term, consider only $t \in (-\delta, \delta)$ with δ small enough so that $e+tk \in \mathrm{int}E$. Then $e_\iota, e_\iota+tk_\iota \in (0, h_\iota)$ for all $\iota \in I$ and $F_\iota(e_\iota+tk_\iota), F_\iota(e_\iota) > F_\iota(0) \geq 0$. Let $\{t_n\}_{n \in \mathbb{N}} \subseteq (-\delta, \delta) \setminus \{0\}$ be a sequence which converges to 0. For all $\iota \in I$

$$\frac{v(F_\iota(e_\iota+t_n k_\iota)) - v(F_\iota(e_\iota))}{t_n} \to v'(F_\iota(e_\iota)) F_\iota'(e_\iota) k_\iota.$$

- $\varphi \geq \psi$ implies $f(\varphi) \geq f(\psi)$,

- $f(\varphi + c) = f(\varphi) + c$.

See Dolecki and Greco [45].

Moreover, for all $n \in \mathbb{N}$, there exists $\bar{e}_{\iota,n}$ between e_ι and $e_\iota + t_n k_\iota$ and $\varphi_{\iota,n}$ between $F_\iota(e_\iota)$ and $F_\iota(e_\iota + t_n k_\iota)$ such that

$$\left| \frac{v(F_\iota(e_\iota + t_n k_\iota)) - v(F_\iota(e_\iota))}{t_n} \right| = \left| \frac{v'(\varphi_{\iota,n})(F_\iota(e_\iota + t_n k_\iota) - F_\iota(e_\iota))}{t_n} \right|$$
$$= \left| \frac{v'(\varphi_{\iota,n}) F'_\iota(\bar{e}_{\iota,n}) t_n k_\iota}{t_n} \right|$$
$$= \left| v'(\varphi_{\iota,n}) F'_\iota(\bar{e}_{\iota,n}) \right| |k_\iota|$$

but notice that $0 \le \bar{e}_{\iota,n} \le \|h\|_{\sup}$ and $0 \le \varphi_{\iota,n} \le \sup_{j \in I} F_j(h_j) < \infty$ (by H.2.i). Therefore, H.4.iii delivers the existence of $r > 0$ such that

$$\left| \frac{v(F_\iota(e_\iota + t_n k_\iota)) - v(F_\iota(e_\iota))}{t_n} \right| \le r \|k\|_{\sup}.$$

The Dominated Convergence Theorem yields

$$\int \frac{v(F_\iota(e_\iota + t_n k_\iota)) - v(F_\iota(e_\iota))}{t_n} d\lambda(\iota) \to \int v'(F_\iota(e_\iota)) F'_\iota(e_\iota) k_\iota d\lambda(\iota).$$

The arbitrary choice of $\{t_n\}_{n \in \mathbb{N}}$ implies

$$\int \frac{v(F_\iota(e_\iota + t k_\iota)) - v(F_\iota(e_\iota))}{t} d\lambda(\iota) \to \int v'(F_\iota(e_\iota)) F'_\iota(e_\iota) k_\iota d\lambda(\iota) \text{ as } t \to 0.$$

Therefore for all i, the function defined for all $t \in (-\delta, \delta)$ by

$$\phi_i(t) = v(F_i(e_i + t k_i)) - \int v(F_\iota(e_\iota + t k_\iota)) d\lambda(\iota)$$

is such that

$$\lim_{t \to 0} \frac{\phi_i(t) - \phi_i(0)}{t} = \lim_{t \to 0} \frac{v(F_i(e_i + t k_i)) - \int v(F_\iota(e_\iota + t k_\iota)) d\lambda(\iota) - \left(v(F_i(e_i)) - \int v(F_\iota(e_\iota)) d\lambda(\iota) \right)}{t}$$
$$= \lim_{t \to 0} \frac{(v(F_i(e_i + t k_i)) - v(F_i(e_i)))}{t} - \int \frac{v(F_\iota(e_\iota + t k_\iota)) - v(F_\iota(e_\iota))}{t} d\lambda(\iota)$$
$$= v'(F_i(e_i)) F'_i(e_i) k_i - \int v'(F_\iota(e_\iota)) F'_\iota(e_\iota) k_\iota d\lambda(\iota)$$

hence it is differentiable in 0 and $\phi'_i(0) = v'(F_i(e_i)) F'_i(e_i) k_i - \int v'(F_\iota(e_\iota)) F'_\iota(e_\iota) k_\iota d\lambda(\iota)$. It follows that $\gamma_i \circ \phi_i$ is differentiable in 0 and

$$\lim_{t \to 0} \frac{\gamma_i \left(v(F_i(e_i + t k_i)) - \int v(F_\iota(e_\iota + t k_\iota)) d\lambda(\iota) \right) - \gamma_i \left(v(F_i(e_i)) - \int v(F_\iota(e_\iota)) d\lambda(\iota) \right)}{t}$$
$$= \gamma'_i(\phi_i(0)) \phi'_i(0)$$
$$= \gamma'_i \left(v(F_i(e_i)) - \int v(F_\iota(e_\iota)) d\lambda(\iota) \right) \left(v'(F_i(e_i)) F'_i(e_i) k_i - \int v'(F_\iota(e_\iota)) F'_\iota(e_\iota) k_\iota d\lambda(\iota) \right).$$

which proves (50).

If $f : L \to \mathbb{R}$ is concave niveloid, then it is Lipschitz and its superdifferential at each point consists of probability charges that are absolutely continuous with respect to λ.

By a chain rule for the Clarke differential (see [39, Theorem 2.3.10]), we have that $f \circ W$ is Lipschitz near e and $\partial(f \circ W)(e) \subseteq \partial f(W(e)) \circ \nabla W(e)$. That is, for all $\mu \in \partial(f \circ W)(e)$ there is $\nu \in \partial f(W(e))$ such that $\mu = \nu \circ \nabla W(e)$. Therefore, for all $k \in L$

$$\mu(k) = \nu(\nabla W(e)(k)) = \int_I \nabla W(e)(k) d\nu$$
$$= \int_I \left[k_i U'_i(e_i) + \gamma'_i \left(v(F_i(e_i)) - \int v(F_\iota(e_\iota)) d\lambda(\iota) \right) \left(v'(F_i(e_i)) F'_i(e_i) k_i - \int v'(F_\iota(e_\iota)) F'_\iota(e_\iota) k_\iota d\lambda(\iota) \right) \right] d\nu(i).$$

If, as assumed *per contra*, e^* is a local maximum of $f \circ W$ on E, then $\partial (f \circ W)(e^*) \ni 0$, and there exists a probability charge $\nu \in \partial f(W(e^*))$ such that $\nu \circ \nabla W(e^*) = 0$, that is, for all $k \in L$

$$\int_I \left[k_i U_i'(e_i^*) + \gamma_i' \left(v(F_i(e_i^*)) - \int v(F_\iota(e_\iota^*)) \, d\lambda(\iota) \right) \left(v'(F_i(e_i^*)) F_i'(e_i^*) k_i - \int v'(F_\iota(e_\iota^*)) F_\iota'(e_\iota^*) k_\iota d\lambda(\iota) \right) \right] d\nu(i) = 0 \tag{51}$$

But, for all $i \in I$, problem (46) is equivalent to

$$\max_{0 \leq y \leq h_i} u_i(F_i(y), y) + \gamma_i(v(F_i(y)) - m^*). \tag{52}$$

Therefore, if (c^*, e^*) is an internal social equilibrium,

- $c_i^* = F_i(e_i^*)$ for all $i \in I$.

- e_i^* is a solution of problem (52) for λ-almost all $i \in I$.

- $m^* = \int v(F_\iota(e_\iota^*)) \, d\lambda(\iota)$.

In particular, first order conditions implied by the second point, see (47) and recall that now γ_i is differentiable, amount to

$$U_i'(e_i^*) + \gamma_i'(v(F_i(e_i^*)) - m^*) v'(F_i(e_i^*)) F_i'(e_i^*) = 0$$

for λ-almost all $i \in I$. Which plugged into (51) delivers, for all $k \in L$

$$\int_I \left[k_i U_i'(e_i^*) + \gamma_i'(v(F_i(e_i^*)) - m^*) \left(v'(F_i(e_i^*)) F_i'(e_i^*) k_i - \int v'(F_\iota(e_\iota^*)) F_\iota'(e_\iota^*) k_\iota d\lambda(\iota) \right) \right] d\nu(i) = 0$$

$$\int_I \left[k_i U_i'(e_i^*) + \gamma_i'(v(F_i(e_i^*)) - m^*) v'(F_i(e_i^*)) F_i'(e_i^*) k_i - \gamma_i'(v(F_i(e_i^*)) - m^*) \int v'(F_\iota(e_\iota^*)) F_\iota'(e_\iota^*) k_\iota d\lambda(\iota) \right] d\nu(i) = 0$$

$$\int_I \left[k_i \left[U_i'(e_i^*) + \gamma_i'(v(F_i(e_i^*)) - m^*) v'(F_i(e_i^*)) F_i'(e_i^*) \right] - \gamma_i'(v(F_i(e_i^*)) - m^*) \int v'(F_\iota(e_\iota^*)) F_\iota'(e_\iota^*) k_\iota d\lambda(\iota) \right] d\nu(i) = 0$$

$$\int_I \left[-\gamma_i'(v(F_i(e_i^*)) - m^*) \int v'(F_\iota(e_\iota^*)) F_\iota'(e_\iota^*) k_\iota d\lambda(\iota) \right] d\nu(i) = 0$$

$$\int v'(F_\iota(e_\iota^*)) F_\iota'(e_\iota^*) k_\iota d\lambda(\iota) \int_I -\gamma_i'(v(F_i(e_i^*)) - m^*) \, d\nu(i) = 0.$$

Since v' and F_ι' are positive and λ is σ-additive, then $\int v'(F_\iota(e_\iota^*)) F_\iota'(e_\iota^*) k_\iota d\lambda(\iota) > 0$ for some $k \in L$ (e.g. $k_\iota = 1$ for all $\iota \in I$) and it must be the case that

$$\int_I -\gamma_i'(v(F_i(e_i^*)) - m^*) \, d\nu(i) = 0$$

which is absurd since γ_i' is bounded away from 0. ∎

Proof of Proposition 12: it is an immediate consequence of Lemma 15 and Proposition 17∎

Proof of Lemma 4. Clearly U is continuous and strictly concave on $[0, \bar{x}_0]$. Therefore $\arg \max_{x \in [0, \bar{x}_0]} U(x)$ is a singleton. The conditions on the directional derivatives provide internality. ∎

Proof of Theorem 7. Set

$$W(x) = U(x) + \gamma(v(x) - m_0^*) + \beta \sum_{s \in S} p_s \left[\gamma(v(\bar{x}_s + R(\bar{x}_0 - x)) - m_s^*) \right]$$

for all $x \in [0, \bar{x}_0]$, and $y_s = \bar{x}_s + R(\bar{x}_0 - x)$ for all $s \in S$.

For all $x^* \in [0, \bar{x}_0)$,

$$
\begin{aligned}
D^+ W(x^*) = \limsup_{h \to 0^+} &\left\{ \frac{U(x^* + h) - U(x^*)}{h} + \frac{\gamma(v(x^* + h) - m_0^*) - \gamma(v(x^*) - m_0^*)}{h} \right. \\
&\left. + \frac{\sum_{s \in S} \beta p_s \left[\gamma(v(\bar{x}_s + R(\bar{x}_0 - x^* - h)) - m_s^*) - \gamma(v(\bar{x}_s + R(\bar{x}_0 - x^*)) - m_s^*) \right]}{h} \right\}
\end{aligned}
$$

$$
\begin{aligned}
= \limsup_{h \to 0^+} &\left\{ \frac{U(x^* + h) - U(x^*)}{h} + \frac{\gamma(v(x^* + h) - m_0^*) - \gamma(v(x^*) - m_0^*)}{h} \right. \\
&\left. + \frac{\sum_{s \in S} \beta p_s \left[\gamma(v(y_s^* - Rh) - m_s^*) - \gamma(v(y_s^*) - m_s^*) \right]}{h} \right\}
\end{aligned}
$$

$$
\begin{aligned}
\leq & \limsup_{h \to 0^+} \frac{U(x^* + h) - U(x^*)}{h} \\
&\dotplus \limsup_{h \to 0^+} \frac{\gamma(v(x^* + h) - m_0^*) - \gamma(v(x^*) - m_0^*)}{h} \\
&\dotplus \beta \sum_{s \in S} p_s \limsup_{h \to 0^+} \frac{\gamma(v(y_s^* - Rh) - m_s^*) - \gamma(v(y_s^*) - m_s^*)}{h}
\end{aligned}
$$

and

$$
\begin{aligned}
D_+ W(x^*) = \liminf_{h \to 0^+} &\left\{ \frac{U(x^* + h) - U(x^*)}{h} + \frac{\gamma(v(x^* + h) - m_0^*) - \gamma(v(x^*) - m_0^*)}{h} \right. \\
&\left. + \frac{\sum_{s \in S} \beta p_s \left[\gamma(v(\bar{x}_s + R(\bar{x}_0 - x^* - h)) - m_s^*) - \gamma(v(\bar{x}_s + R(\bar{x}_0 - x^*)) - m_s^*) \right]}{h} \right\}
\end{aligned}
$$

$$
\begin{aligned}
= \liminf_{h \to 0^+} &\left\{ \frac{U(x^* + h) - U(x^*)}{h} + \frac{\gamma(v(x^* + h) - m_0^*) - \gamma(v(x^*) - m_0^*)}{h} \right. \\
&\left. + \frac{\sum_{s \in S} \beta p_s \left[\gamma(v(y_s^* - Rh) - m_s^*) - \gamma(v(y_s^*) - m_s^*) \right]}{h} \right\}
\end{aligned}
$$

$$
\begin{aligned}
\geq & \liminf_{h \to 0^+} \frac{U(x^* + h) - U(x^*)}{h} \\
&+ \liminf_{h \to 0^+} \frac{\gamma(v(x^* + h) - m_0^*) - \gamma(v(x^*) - m_0^*)}{h} \\
&+ \beta \sum_{s \in S} p_s \liminf_{h \to 0^+} \frac{\gamma(v(y_s^* - Rh) - m_s^*) - \gamma(v(y_s^*) - m_s^*)}{h}.
\end{aligned}
$$

Analogously, for all $x^* \in (0, \bar{x}_0]$,

$$
\begin{aligned}
D^- W(x^*) = \limsup_{h \to 0^-} &\left\{ \frac{U(x^* + h) - U(x^*)}{h} + \frac{\gamma(v(x^* + h) - m_0^*) - \gamma(v(x^*) - m_0^*)}{h} \right. \\
&\left. + \frac{\sum_{s \in S} \beta p_s \left[\gamma(v(\bar{x}_s + R(\bar{x}_0 - x^* - h)) - m_s^*) - \gamma(v(\bar{x}_s + R(\bar{x}_0 - x^*)) - m_s^*) \right]}{h} \right\}
\end{aligned}
$$

$$
\begin{aligned}
= \limsup_{h \to 0^-} &\left\{ \frac{U(x^* + h) - U(x^*)}{h} + \frac{\gamma(v(x^* + h) - m_0^*) - \gamma(v(x^*) - m_0^*)}{h} \right. \\
&\left. + \frac{\sum_{s \in S} \beta p_s \left[\gamma(v(y_s^* - Rh) - m_s^*) - \gamma(v(y_s^*) - m_s^*) \right]}{h} \right\}
\end{aligned}
$$

$$
\begin{aligned}
\leq & \limsup_{h \to 0^-} \frac{U(x^* + h) - U(x^*)}{h} \\
&\dotplus \limsup_{h \to 0^-} \frac{\gamma(v(x^* + h) - m_0^*) - \gamma(v(x^*) - m_0^*)}{h} \\
&\dotplus \beta \sum_{s \in S} p_s \limsup_{h \to 0^-} \frac{\gamma(v(y_s^* - Rh) - m_s^*) - \gamma(v(y_s^*) - m_s^*)}{h}
\end{aligned}
$$

and

$$D_- W\left(x^*\right) = \liminf_{h \to 0^-} \left\{ \frac{U\left(x^* + h\right) - U\left(x^*\right)}{h} + \frac{\gamma\left(v\left(x^* + h\right) - m_0^*\right) - \gamma\left(v\left(x^*\right) - m_0^*\right)}{h} \right.$$
$$\left. + \frac{\sum_{s \in S} \beta p_s \left[\gamma\left(v(\bar{x}_s + R(\bar{x}_0 - x^* - h)) - m_s^*\right) - \gamma\left(v(\bar{x}_s + R(\bar{x}_0 - x^*) - m_s^*\right)\right]}{h} \right\}$$

$$= \liminf_{h \to 0^-} \left\{ \frac{U\left(x^* + h\right) - U\left(x^*\right)}{h} + \frac{\gamma\left(v\left(x^* + h\right) - m_0^*\right) - \gamma\left(v\left(x^*\right) - m_0^*\right)}{h} \right.$$
$$\left. + \frac{\sum_{s \in S} \beta p_s \left[\gamma\left(v(y_s^* - Rh) - m_s^*\right) - \gamma\left(v\left(y_s^*\right) - m_s^*\right)\right]}{h} \right\}$$

$$\geq \liminf_{h \to 0^-} \frac{U\left(x^* + h\right) - U\left(x^*\right)}{h}$$
$$+ \liminf_{h \to 0^-} \frac{\gamma\left(v\left(x^* + h\right) - m_0^*\right) - \gamma\left(v\left(x^*\right) - m_0^*\right)}{h}$$
$$+ \beta \sum_{s \in S} p_s \liminf_{h \to 0^-} \frac{\gamma\left(v(y_s^* - Rh) - m_s^*\right) - \gamma\left(v\left(y_s^*\right) - m_s^*\right)}{h}.$$

(i) Consider any symmetric consumption profile, where all agents consume the same amount $x^* \in [0, \bar{x}_0]$ in the first period (i.e. $c_i^* = x^*$ for all $i \in I$), and $y_s^* = \bar{x}_s + R(\bar{x}_0 - x^*)$ in each state in the second period. Then

$$m_0^*\left(c^*\right) = \int_I v\left(x^*\right) d\lambda\left(\iota\right) = v\left(x^*\right)$$
$$m_s^*\left(c^*\right) = \int_I v\left(\bar{x}_s + R\left(\bar{x}_0 - x^*\right)\right) d\lambda\left(\iota\right) = v(y_s^*) \quad \forall s \in S.$$

For $x^* \in [0, \bar{x}_0)$

$$D_+ W\left(x^*\right) \geq U_+'\left(x^*\right) + \liminf_{h \to 0^+} \frac{\gamma\left(v\left(x^* + h\right) - v\left(x^*\right)\right) - \gamma\left(v\left(x^*\right) - v\left(x^*\right)\right)}{h}$$
$$+ \beta \sum_{s \in S} p_s \liminf_{h \to 0^+} \frac{\gamma\left(v(y_s^* - Rh) - v\left(y_s^*\right)\right) - \gamma\left(v\left(y_s^*\right) - v\left(y_s^*\right)\right)}{h}$$

$$= U_+'\left(x^*\right) + \liminf_{h \to 0^+} \frac{\gamma\left(v\left(x^* + h\right) - v\left(x^*\right)\right) - \gamma\left(0\right)}{h}$$
$$+ \beta \sum_{s \in S} p_s \sup_{\delta > 0} \inf_{h \in (0, \delta)} \frac{\gamma\left(v(y_s^* - Rh)) - v\left(y_s^*\right)\right) - \gamma\left(v\left(y_s^*\right) - v\left(y_s^*\right)\right)}{h}$$

since $\gamma_{|(-\infty, 0]} \equiv 0$, then

$$D_+ W\left(x^*\right) \geq U_+'\left(x^*\right) + \liminf_{h \to 0^+} \frac{\gamma\left(v\left(x^* + h\right) - v\left(x^*\right)\right) - \gamma\left(0\right)}{h}.$$

Moreover $v_0 : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$v_0\left(h\right) = v\left(x^* + h\right) - v\left(x^*\right) \quad \forall h \in [0, +\infty)$$

is concave, strictly increasing and continuous with

$$v_0\left(0\right) = 0,$$
$$\left(v_0\right)_+'\left(0\right) = \lim_{h \to 0^+} \frac{v\left(x^* + h\right) - v\left(x^*\right)}{h} = v_+'\left(x^*\right)$$

thus

$$D_+W\left(x^*\right) \geq U'_+\left(x^*\right) + \liminf_{h\to 0^+} \frac{\gamma\left(v_0\left(h\right)\right) - \gamma\left(v_0\left(0\right)\right)}{h}$$
$$= U'_+\left(x^*\right) + D_+\left(\gamma \circ v_0\right)\left(0\right).$$

The assumption $D_+\gamma\left(0\right) > 0$ allows to apply a chain rule for Dini derivatives so that

$$D_+W\left(x^*\right) \geq U'_+\left(x^*\right) + \left(v_0\right)'_+\left(0\right) D_+\gamma\left(v_0\left(0\right)\right) = U'_+\left(x^*\right) + v'_+\left(x^*\right) D_+\gamma\left(0\right).$$

Analogously, for $x^* \in \left(0, \bar{x}_0\right]$

$$D^-W\left(x^*\right) \leq U'_-\left(x^*\right) \dotplus \limsup_{h\to 0^-} \frac{\gamma\left(v\left(x^* + h\right) - v\left(x^*\right)\right) - \gamma\left(0\right)}{h}$$
$$\dotplus \beta \sum_{s\in S} p_s \limsup_{h\to 0^-} \frac{\gamma\left(v(y_s^* - Rh) - v\left(y_s^*\right)\right) - \gamma\left(0\right)}{h}$$
$$= U'_-\left(x^*\right) \dotplus \inf_{\delta>0} \sup_{h(0,\delta)} \frac{\gamma\left(v\left(x^* - h\right) - v\left(x^*\right)\right) - \gamma\left(0\right)}{-h}$$
$$\dotplus \beta \sum_{s\in S} p_s \inf_{\delta>0} \sup_{h(0,\delta)} \frac{\gamma\left(v(y_s^* + Rh) - v\left(y_s^*\right)\right) - \gamma\left(0\right)}{-h}$$

since $\gamma_{|(-\infty,0]} \equiv 0$, then

$$D^-W\left(x^*\right) \leq U'_-\left(x^*\right) \dotplus \left(-\beta \sum_{s\in S} p_s \sup_{\delta>0} \inf_{h(0,\delta)} \frac{\gamma\left(v(y_s^* + Rh) - v\left(y_s^*\right)\right) - \gamma\left(0\right)}{h}\right)$$
$$= U'_-\left(x^*\right) \dotminus \beta \sum_{s\in S} p_s \liminf_{h\to 0^+} \frac{\gamma\left(v(y_s^* + Rh) - v\left(y_s^*\right)\right) - \gamma\left(0\right)}{h}.$$

Moreover, for all $s \in S$, $v_s : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$v_s\left(h\right) = v\left(y_s^* + Rh\right) - v\left(y_s^*\right) \quad \forall h \in \left[0, +\infty\right)$$

is concave, strictly increasing and continuous with

$$v_s\left(0\right) = 0,$$
$$\left(v_s\right)'_+\left(0\right) = \lim_{h\to 0^+} \frac{v\left(y_s^* + Rh\right) - v\left(y_s^*\right)}{h} = \lim_{h\to 0^+} \frac{v\left(y_s^* + Rh\right) - v\left(y_s^*\right)}{Rh} R = Rv'_+\left(y_s^*\right)$$

thus

$$D^-W\left(x^*\right) \leq U'_-\left(x^*\right) \dotminus \beta \sum_{s\in S} p_s \liminf_{h\to 0^+} \frac{\gamma\left(v_s(h)\right) - \gamma\left(v_s\left(0\right)\right)}{h}$$
$$= U'_-\left(x^*\right) \dotminus \beta \sum_{s\in S} p_s D_+\left(\gamma \circ v_s\right)\left(0\right).$$

The assumption $D_+\gamma\left(0\right) > 0$ allows to apply a chain rule for Dini derivatives so that

$$D^-W\left(x^*\right) \leq U'_-\left(x^*\right) \dotminus \beta \sum_{s\in S} p_s \left(v_s\right)'_+\left(0\right) D_+\gamma\left(v_s\left(0\right)\right) = U'_-\left(x^*\right) \dotminus \beta R \sum_{s\in S} p_s v'_+\left(y_s^*\right) D_+\gamma\left(0\right).$$

Summing up:

$$D_+W\left(x^*\right) \geq U'_+\left(x^*\right) + v'_+\left(x^*\right) D_+\gamma\left(0\right) \quad \forall x^* \in \left[0, \bar{x}_0\right) \text{ and}$$
$$D^-W\left(x^*\right) \leq U'_-\left(x^*\right) \dotminus \beta R \sum_{s\in S} p_s v'_+\left(y_s^*\right) D_+\gamma\left(0\right) \quad \forall x^* \in \left(0, \bar{x}_0\right].$$

- If $x^* \in (0, \bar{x}_0)$, then, since u is differentiable on $(0, +\infty)$, U is differentiable at x^* and $U'_+(x^*) = U'_-(x^*) = U'(x^*)$, thus

  - either $U'(x^*) \geq 0$, then $D_+W(x^*) > 0$ and x^* is not a maximizer.
  - either $U'(x^*) < 0$, then $D^-W(x^*) < 0$ and x^* is not a maximizer.

- If $x^* = 0$ then $U'_+(0) > 0$ and $D_+W(x^*) > 0$, thus x^* is not a maximizer.

- If $x^* = \bar{x}_0$ then $U'_-(\bar{x}_0) < 0$ and $D^-W(x^*) < 0$, thus x^* is not a maximizer.

(ii) Notice that if $c^* \in L$ is a social equilibrium, then, setting $m_0^* = \int_I v(c_\iota)\, d\lambda(\iota)$ and $m_s^* = \int_I v(\bar{x}_s + R(\bar{x}_0 - c_\iota))d\lambda(\iota)$ for all $s \in S$, c_i^* is a solution of problem

$$\max_{x \in [0, \bar{x}_0]} U(x) + \gamma(v(x) - m_0^*) + \beta \sum_{s \in S} p_s [\gamma(v(\bar{x}_s + R(\bar{x}_0 - x)) - m_s^*)] \tag{53}$$

for λ-almost all $i \in I$.

Let $c^* : I \to \mathbb{R}$ be an asymmetric social equilibrium and $I^* \in \Lambda$ be such that $\lambda(I^*) = 1$ and c_i^* is a solution of problem (53) for all $i \in I^*$.

There is at least one agent, call him $i_0 \in I^*$, such that $v(c_{i_0}^*) < m_0^*$.[63] Analogously, for all $s \in S$ there exists $i_s \in I^*$, such that $v(\bar{x}_s + R(\bar{x}_0 - c_{i_s}^*)) < m_s^*$ for all $s \in S$.

Suppose agent i_1 is such that $c_{i_1}^* = \max_{s \in S} c_{i_s}^*$. Then,

$$v(\bar{x}_s + R(\bar{x}_0 - c_{i_1}^*)) \leq v(\bar{x}_s + R(\bar{x}_0 - c_{i_s}^*)) < m_s^* \quad \forall s \in S.$$

Since v is concave, by the Jensen inequality:

$$v(c_{i_0}^*) < m_0^* < v(c_{i_1}^*)$$
$$v(\bar{x}_s + R(\bar{x}_0 - c_{i_1}^*)) < m_s^* < v(\bar{x}_s + R(\bar{x}_0 - c_{i_0}^*)) \quad \forall s \in S.$$

In fact,

$$v(c_{i_0}^*) < m_0^* = \int v(c^*(\iota))\, d\lambda(\iota) \leq v\left(\int c^*(\iota)\, d\lambda(\iota)\right) \implies c_{i_0}^* < \int c^*(\iota)\, d\lambda(\iota)$$

but $v_s : x \mapsto v(\bar{x}_s + R(\bar{x}_0 - x))$ is concave for all $s \in S$ (since v is concave) and

$$v(\bar{x}_s + R(\bar{x}_0 - c_{i_0}^*)) > v\left(\bar{x}_s + R\left(\bar{x}_0 - \int c^*(\iota)\, d\lambda(\iota)\right)\right) = v_s\left(\int c^*(\iota)\, d\lambda(\iota)\right)$$
$$\geq \int v_s(c^*(\iota))d\lambda(\iota) = \int v(\bar{x}_s + R(\bar{x}_0 - c_\iota^*))d\lambda(\iota) = m_s^*$$

that is $m_s^* < v(\bar{x}_s + R(\bar{x}_0 - c_{i_0}^*))$ for all $s \in S$. Analogously, for any $s \in S$

$$v(\bar{x}_s + R(\bar{x}_0 - c_{i_1}^*)) < m_s^* = \int v_s(c^*(\iota))d\lambda(\iota)$$
$$\leq v_s\left(\int c^*(\iota)\, d\lambda(\iota)\right) = v\left(\bar{x}_s + R\left(\bar{x}_0 - \int c^*(\iota)\, d\lambda(\iota)\right)\right)$$

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[63]Assume per contra $v(c_i^*) \geq m_0^*$ for all $i \in I^*$, then $(v \circ c^* - \int_I v(c_\iota^*)\, d\lambda(\iota)) \geq 0$ λ-a.e. and

$$\int_I \left(v(c_i^*) - \int_I v(c_\iota^*)\, d\lambda(\iota)\right) d\lambda(i) = 0$$

therefore $v \circ c^* = \int_I v(c_\iota^*)\, d\lambda(\iota)$ λ-a.e. and $c^* = v^{-1}\left(\int_I v(c_\iota^*)\, d\lambda(\iota)\right)$ λ-a.e., which is absurd since c^* is asymmetric.

then $c^*_{i_1} > \int c^*(\iota)\, d\lambda(\iota))$ and

$$v\left(c^*_{i_1}\right) > v\left(\int c^*(\iota)\, d\lambda(\iota))\right) \geq \int v(c^*(\iota))\, d\lambda(\iota) = m^*_0.$$

Summing up, there exists $i_0, i_1 \in I^*$ such that $c^*_{i_0}$ and $c^*_{i_1}$ are maximizers for W on $[0, \bar{x}_0]$, and

$$v\left(c^*_{i_0}\right) < m^*_0 < v\left(c^*_{i_1}\right)$$
$$v(\bar{x}_s + R(\bar{x}_0 - c^*_{i_1})) < m^*_s < v(\bar{x}_s + R(\bar{x}_0 - c^*_{i_0})) \quad \forall s \in S.$$

In particular $0 \leq c^*_{i_0} < c^*_{i_1} \leq \bar{x}_0$. Since v is continuous and increasing, there is $\varepsilon > 0$ small enough so that

$$0 \leq c^*_{i_0} < c^*_{i_0} + \varepsilon < c^*_{i_1} - \varepsilon < c^*_{i_1} \leq \bar{x}_0, \tag{54}$$

$$v(x) < m^*_0 \text{ and } m^*_s < v(\bar{x}_s + R(\bar{x}_0 - x)) \quad \forall s \in S, \forall x \in \left[0, c^*_{i_0} + \varepsilon\right), \tag{55}$$

$$m^*_0 < v(x) \text{ and } v(\bar{x}_s + R(\bar{x}_0 - x)) < m^*_s \quad \forall s \in S, \forall x \in \left(c^*_{i_1} - \varepsilon, \bar{x}_0\right]. \tag{56}$$

The first order conditions at $c^*_{i_0}$ guarantee that

$$0 \geq D_+ W\left(c^*_{i_0}\right) \geq \liminf_{h \to 0^+} \frac{U\left(c^*_{i_0} + h\right) - U\left(c^*_{i_0}\right)}{h}$$
$$+ \liminf_{h \to 0^+} \frac{\gamma\left(v\left(c^*_{i_0} + h\right) - m^*_0\right) - \gamma\left(v\left(c^*_{i_0}\right) - m^*_0\right)}{h}$$
$$+ \beta \sum_{s \in S} p_s \liminf_{h \to 0^+} \frac{\gamma\left(v\left(\bar{x}_s + R(\bar{x}_0 - c^*_{i_0}) - Rh\right) - m^*_s\right) - \gamma\left(v\left(\bar{x}_s + R(\bar{x}_0 - c^*_{i_0})\right) - m^*_s\right)}{h}$$
$$= U'_+\left(c^*_{i_0}\right) + \liminf_{h \to 0^+} \frac{\gamma\left(v\left(c^*_{i_0} + h\right) - m^*_0\right) - \gamma\left(v\left(c^*_{i_0}\right) - m^*_0\right)}{h}$$

since the third summand is null because of $\gamma_{|[0,+\infty)} \equiv 0$, equation (55), and the observation that

$$\liminf_{h \to 0^+} \frac{\gamma\left(v\left(\bar{x}_s + R(\bar{x}_0 - c^*_{i_0}) - Rh\right) - m^*_s\right) - \gamma\left(v\left(\bar{x}_s + R(\bar{x}_0 - c^*_{i_0})\right) - m^*_s\right)}{h}$$
$$= \lim_{\varepsilon \to 0^+} \inf_{h \in (0,\varepsilon)} \frac{\gamma\left(v\left(\bar{x}_s + R\left(\bar{x}_0 - (c^*_{i_0} + h)\right)\right) - m^*_s\right) - \gamma\left(v\left(\bar{x}_s + R(\bar{x}_0 - c^*_{i_0})\right) - m^*_s\right)}{h}.$$

Consider the function $v_0 : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$v_0(h) = v\left(c^*_{i_0} + h\right) - m^*_0 \quad \forall h \in [0, +\infty)$$

v_0 is concave, strictly increasing and continuous with

$$v_0(0) = v\left(c^*_{i_0}\right) - m^*_0,$$
$$(v_0)'_+(0) = \lim_{h \to 0^+} \frac{v\left(c^*_{i_0} + h\right) - m^*_0 - v\left(c^*_{i_0}\right) + m^*_0}{h} = v'_+\left(c^*_{i_0}\right)$$

and the condition $D_+ \gamma_{|(-\infty,0)} > 0$, equation (55), and a suitable chain rule deliver

$$0 \geq D_+ W\left(c^*_{i_0}\right) \geq U'_+\left(c^*_{i_0}\right) + \liminf_{h \to 0^+} \frac{\gamma\left(v_0(h)\right) - \gamma\left(v_0(0)\right)}{h}$$
$$= U'_+\left(c^*_{i_0}\right) + v'_+\left(c^*_{i_0}\right) D_+ \gamma\left(v\left(c^*_{i_0}\right) - m^*_0\right)$$

that is
$$U'_+ \left(c^*_{i_0}\right) \le -v'_+ \left(c^*_{i_0}\right) D_+\gamma \left(v \left(c^*_{i_0}\right) - m^*_0\right) < 0. \tag{57}$$

Analogously, the first order conditions at $c^*_{i_1}$ guarantee that

$$0 \le D^-W \left(c^*_{i_1}\right) \le$$

$$\le \limsup_{h\to 0^-} \frac{U \left(c^*_{i_1} + h\right) - U \left(c^*_{i_1}\right)}{h}$$

$$\dotplus \limsup_{h\to 0^-} \frac{\gamma \left(v \left(c^*_{i_1} + h\right) - m^*_0\right) - \gamma \left(v \left(c^*_{i_1}\right) - m^*_0\right)}{h}$$

$$\dotplus \beta \sum_{s\in S} p_s \limsup_{h\to 0^-} \frac{\gamma \left(v \left(\left(\bar{x}_s + R \left(\bar{x}_0 - c^*_{i_1}\right)\right) - Rh\right) - m^*_s\right) - \gamma \left(v \left(\bar{x}_s + R \left(\bar{x}_0 - c^*_{i_1}\right)\right) - m^*_s\right)}{h}$$

$$= U'_- \left(c^*_{i_1}\right) \dotplus \beta \sum_{s\in S} p_s \inf_{\delta>0} \sup_{h\in(0,\delta)} \frac{\gamma \left(v \left(\left(\bar{x}_s + R \left(\bar{x}_0 - c^*_{i_1}\right)\right) + Rh\right) - m^*_s\right) - \gamma \left(v \left(\bar{x}_s + R \left(\bar{x}_0 - c^*_{i_1}\right)\right) - m^*_s\right)}{-h}$$

since the second summand is null because of $\gamma_{|[0,+\infty)} \equiv 0$ and equation (56). Consider, for all $s \in S$ the function $v_s : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$v_s \left(h\right) = v \left(\left(\bar{x}_s + R \left(\bar{x}_0 - c^*_{i_1}\right)\right) + Rh\right) - m^*_s \quad \forall h \in [0, +\infty)$$

is concave, strictly increasing and continuous with

$$v_s \left(0\right) = v \left(\bar{x}_s + R \left(\bar{x}_0 - c^*_{i_1}\right)\right) - m^*_s,$$

$$\left(v_s\right)'_+ \left(0\right) = \lim_{h\to 0^+} \frac{v \left(\left(\bar{x}_s + R \left(\bar{x}_0 - c^*_{i_1}\right)\right) + Rh\right) - m^*_s - v \left(\bar{x}_s + R \left(\bar{x}_0 - c^*_{i_1}\right)\right) + m^*_s}{h}$$

$$= \lim_{h\to 0^+} \frac{v \left(\left(\bar{x}_s + R \left(\bar{x}_0 - c^*_{i_1}\right)\right) + Rh\right) - v \left(\bar{x}_s + R \left(\bar{x}_0 - c^*_{i_1}\right)\right)}{Rh} R$$

$$= Rv'_+ \left(\bar{x}_s + R \left(\bar{x}_0 - c^*_{i_1}\right)\right)$$

equation (56) implies $v_s \left(0\right) = v(\bar{x}_s + R(\bar{x}_0 - c^*_{i_1}) - m^*_s < 0$, then $D_+\gamma_{|(-\infty,0)} > 0$ and a suitable chain rule deliver

$$0 \le D^-W \left(c^*_{i_1}\right) \le U'_- \left(c^*_{i_1}\right) \dotplus \beta \sum_{s\in S} p_s \inf_{\delta>0} \sup_{h\in(0,\delta)} \frac{\gamma \left(v_s \left(h\right)\right) - \gamma \left(v_s \left(0\right)\right)}{-h}$$

$$= U'_- \left(c^*_{i_1}\right) \dotminus \beta \sum_{s\in S} p_s \sup_{\delta>0} \inf_{h\in(0,\delta)} \frac{\gamma \left(v_s \left(h\right)\right) - \gamma \left(v_s \left(0\right)\right)}{h}$$

$$= U'_- \left(c^*_{i_1}\right) \dotminus \beta \sum_{s\in S} p_s \liminf_{h\to 0^+} \frac{\gamma \left(v_s \left(h\right)\right) - \gamma \left(v_s \left(0\right)\right)}{h}$$

$$= U'_- \left(c^*_{i_1}\right) \dotminus \beta \sum_{s\in S} p_s \left(v_s\right)'_+ \left(0\right) D_+\gamma \left(v_s \left(0\right)\right)$$

$$= U'_- \left(c^*_{i_1}\right) \dotminus \beta R \sum_{s\in S} p_s v'_+ \left(\bar{x}_s + R \left(\bar{x}_0 - c^*_{i_1}\right)\right) D_+\gamma \left(v_s \left(0\right)\right)$$

that is

$$U'_- \left(c^*_{i_1}\right) \ge \beta R \sum_{s\in S} p_s v'_+ \left(\bar{x}_s + R \left(\bar{x}_0 - c^*_{i_1}\right)\right) D_+\gamma \left(v_s \left(0\right)\right) > 0. \tag{58}$$

Denoting by $\hat{c}$ the unique (internal) asocial equilibrium it follows, from the differentiability of u on $(0, +\infty)$, that

$$U'_+ \left(\hat{c}\right) = U'_- \left(\hat{c}\right) = U' \left(\hat{c}\right) = 0.$$

Thus, since U'_+ and U'_- are decreasing, from (57) and (58) it follows that

$$U'_+\left(c^*_{i_0}\right) < 0 = U'_+\left(\hat{c}\right) \Rightarrow c^*_{i_0} > \hat{c}$$
$$U'_-\left(c^*_{i_1}\right) > 0 = U'_-\left(\hat{c}\right) \Rightarrow c^*_{i_1} < \hat{c}$$

and $c^*_{i_0} > c^*_{i_1}$ which contradicts (54). Therefore c^* is not a social equilibrium.

Finally suppose $D^-\gamma\left(0\right) = 0$ and $c^* : I \to \mathbb{R}$ is a symmetric equilibrium with $c^*_i = x^*$ for λ-almost all $i \in I$. Then

$$0 \geq D_+W\left(x^*\right) \geq U'_+\left(x^*\right) + \sup_{\delta>0} \inf_{h\in(0,\delta)} \frac{\gamma\left(v\left(x^* + h\right) - v\left(x^*\right)\right) - \gamma\left(0\right)}{h}$$
$$+ \beta \sum_{s\in S} p_s \sup_{\delta>0} \inf_{h\in(0,\delta)} \frac{\gamma\left(v(y^*_s - Rh)\right) - v\left(y^*_s\right)\right) - \gamma\left(0\right)}{h}$$

if $x^* \in [0, \bar{x}_0)$ and

$$0 \leq D^-W\left(x^*\right) \leq U'_-\left(x^*\right) \dotplus \inf_{\delta>0} \sup_{h(0,\delta)} \frac{\gamma\left(v\left(x^* - h\right) - v\left(x^*\right)\right) - \gamma\left(0\right)}{-h}$$
$$\dotplus \beta \sum_{s\in S} p_s \inf_{\delta>0} \sup_{h(0,\delta)} \frac{\gamma\left(v(y^*_s + Rh)\right) - v\left(y^*_s\right)\right) - \gamma\left(0\right)}{-h}$$

if $x^* \in (0, \bar{x}_0]$. Then $\gamma_{|[0,+\infty)} \equiv 0$ delivers

$$0 \geq D_+W\left(x^*\right) \geq U'_+\left(x^*\right) + \beta \sum_{s\in S} p_s \sup_{\delta>0} \inf_{h\in(0,\delta)} \frac{\gamma\left(v(y^*_s - Rh)\right) - v\left(y^*_s\right)\right) - \gamma\left(0\right)}{h}$$
$$= U'_+\left(x^*\right) + \beta \sum_{s\in S} p_s \sup_{\delta>0} \inf_{h\in(0,\delta)} -\frac{\gamma\left(v(y^*_s + R\left(-h\right))\right) - v\left(y^*_s\right)\right) - \gamma\left(0\right)}{\left(-h\right)}$$
$$= U'_+\left(x^*\right) - \beta \sum_{s\in S} p_s \inf_{\delta>0} \sup_{h\in(0,\delta)} \frac{\gamma\left(v(y^*_s + R\left(-h\right))\right) - v\left(y^*_s\right)\right) - \gamma\left(0\right)}{\left(-h\right)}$$
$$= U'_+\left(x^*\right) - \beta \sum_{s\in S} p_s \limsup_{h\to 0^-} \frac{\gamma\left(v(y^*_s + Rh)\right) - v\left(y^*_s\right)\right) - \gamma\left(0\right)}{h}$$

if $x^* \in [0, \bar{x}_0)$, and

$$0 \leq D^-W\left(x^*\right) \leq U'_-\left(x^*\right) \dotplus \inf_{\delta>0} \sup_{h(0,\delta)} \frac{\gamma\left(v\left(x^* - h\right) - v\left(x^*\right)\right) - \gamma\left(0\right)}{-h}$$
$$= U'_-\left(x^*\right) \dotplus \limsup_{h\to 0^-} \frac{\gamma\left(v\left(x^* + h\right) - v\left(x^*\right)\right) - \gamma\left(0\right)}{h}$$

if $x^* \in (0, \bar{x}_0]$.

A suitable chain rule delivers

$$0 \geq D_+W\left(x^*\right) \geq U'_+\left(x^*\right) - \beta R \sum_{s\in S} p_s v'_-\left(y^*_s\right) D^-\gamma\left(0\right)$$

that is, $0 \geq D_+W\left(x^*\right) \geq U'_+\left(x^*\right)$ if $x^* \in [0, \bar{x}_0)$, and

$$0 \leq D^-W\left(x^*\right) \leq U'_-\left(x^*\right) \dotplus v'_-\left(x^*\right) D^-\gamma\left(0\right)$$

that is, $0 \leq D^-W\left(x^*\right) \leq U'_-\left(x^*\right)$ if $x^* \in (0, \bar{x}_0]$. Therefore x^* is a maximizer for U. $\blacksquare$

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