ISSN 2279-9362

Collegio Carlo Alberto

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No. 522
December 2017

Carlo Alberto Notebooks
www.carloalberto.org/research/working-papers

UNIFORM MOMENT PROPAGATION FOR THE BECKER-DÖRING EQUATION

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Abstract. We show uniform-in-time propagation of algebraic and stretched exponential moments for the Becker-Döring equations. Our proof is based upon a suitable use of the maximum principle together with known rates of convergence to equilibrium.

Contents

1. Introduction

In this note we consider the Becker-Döring equations

$$\frac{\mathrm{d}}{\mathrm{d}t}c_i(t) = W_{i-1}(t) - W_i(t), \qquad i \in \mathbb{N} \setminus \{1\}\,, \tag{1.1a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}c_1(t) = -W_1(t) - \sum_{k=1}^{\infty} W_k(t), \tag{1.1b}$$

where

$$W_i(t) := a_i\, c_1(t)c_i(t) - b_{i+1}\, c_{i+1}(t) \qquad i \in \mathbb{N}. \tag{1.2}$$

The unknowns here are the functions $\boldsymbol{c}(t) = (c_i(t))_{i\geqslant 1}$ which depend on time $t \geqslant 0$ and where, for each $i \in \mathbb{N}$, $c_i(t)$ represents the density of clusters of size i at time $t \geqslant 0$ (this is, clusters composed of exactly i individual particles). The nonnegative numbers a_i, b_i denote respectively the coagulation and fragmentation coefficients. These equations are a model for the dynamics of cluster growth in which clusters can only gain or shed one particle; that is, the only reactions taking place are

$$\{i\} + \{1\} \underset{b_{i+1}}{\overset{a_i}{\rightleftharpoons}} \{i+1\},$$

where $\{i\}$ represents the concentration of clusters of size i. The quantity W_i then represents the net rate of this reaction, obtained by standard mass-action dynamics. It is a well-accepted model for the kinetics of first-order phase transitions, applicable to a wide variety of phenomena such as crystallisation, vapor condensation, aggregation of lipids or phase separation in alloys. The model is traced back to Becker and Döring (1935), and the basis of its mathematical theory was set in Ball et al. (1986); Ball and Carr (1988). There have been a number of works on the long-time behaviour of solutions, which is especially interesting since it exhibits phase-change phenomena, metastability, and fast relaxation to equilibrium depending on the regime one is considering. We mention here the works by Penrose (1997, 1989); Velázquez (1998); Jabin and Niethammer (2003); Niethammer (2008); Cañizo and Lods (2013); Cañizo et al. (2017); Murray and Pego (2016b,a); Schlichting (2016), leaving out many relevant ones. We direct the reader to the references in the aforementioned works for a more complete picture, and to the survey paper by Slemrod (2000).

Despite the amount of works devoted to the model, it seems to us that the question of propagation of moments has not been fully answered, and it is our purpose to fill that gap in this paper. The basic question we address is the following: if $\sum_{i=1}^{\infty} i^k c_i(0) < +\infty$ is finite for some $k > 1$, is it true that $\sum_{i=1}^{\infty} i^k c_i(t) \leqslant C$ for some $C > 0$ and all $t \geqslant 0$? The answer is that this is indeed true for subcritical solutions (but of course is not true for supercritical solutions); we do not give an answer for exactly critical solutions (those with density equal to the critical one).

Before describing our results with more detail we need to set some notation and give some background on the asymptotic behaviour of equation (1.1).

1.1. **A quick summary on asymptotic behaviour.** Equation (1.1) can be written, in weak form, as

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{\infty} c_i(t)\phi_i = \sum_{i=1}^{\infty} W_i(t)(\phi_{i+1} - \phi_i - \phi_1), \tag{1.3}$$

for all slowly growing sequences $(\phi_i)_{i\geqslant 1}$. In particular, taking $\phi_i = i$, one sees that the *density* of the solution, defined by

$$\varrho := \sum_{i=1}^{\infty} i c_i(0) = \sum_{i=1}^{\infty} i c_i(t), \tag{1.4}$$

is formally conserved under time evolution. Defining the *detailed balance coefficients* Q_i recursively by

$$Q_1 = 1, \quad Q_{i+1} = \frac{a_i}{b_{i+1}} Q_i \quad i \in \mathbb{N} \tag{1.5}$$

one can see that any sequence of the form $(Q_i z^i)_{i\geqslant 1}$ is formally an equilibrium of (1.1). However, such a sequence may not have a finite density. The largest $z_s \geqslant 0$ (possibly $z_s = +\infty$) for which

$$\sum_{i=1}^{\infty} i Q_i z^i < +\infty \qquad \text{for all } 0 \leqslant z < z_s \tag{1.6}$$

is called the *critical monomer density*, or sometimes the monomer saturation density (alternatively, z_s is the radius of convergence of the power series with coefficients iQ_i). The *critical density* (or, again, saturation density) is then defined by

$$\varrho_s := \sum_{i=1}^{\infty} i Q_i z_s^i \in [0, +\infty].$$

This critical density plays a fundamental role in the long-time behaviour of solutions to (1.1): it was proved in Ball et al. (1986) and Ball and Carr (1988) that any solution with density $\varrho > \varrho_s$ will converge (in a weak sense) to the only equilibrium with density ϱ_s, with the excess mass $\varrho - \varrho_s$ becoming concentrated in larger and larger clusters as time passes. In contrast, any solution with initial density $\varrho \leqslant \varrho_s$ will converge (strongly) as

$t \to \infty$ to an equilibrium solution with its same density ϱ. We focus here on the so-called *subcritical solutions* for which $\varrho < \varrho_s$, which converge to the equilibrium $\boldsymbol{\mathcal{Q}} := (\mathcal{Q}_i)_{i \geqslant 1}$ given by

$$\mathcal{Q}_i = Q_i \overline{z}^i, \qquad i \geqslant 1,$$

where $\overline{z} \in [0, z_s)$ is the unique number such that $\varrho = \sum_{i=1}^{\infty} Q_i \overline{z}^i$. The rate of convergence to this equilibrium in exponentially weighted $\ell_1(\mathbb{N})$ norms was studied in Jabin and Niethammer (2003) and subsequently improved in Cañizo and Lods (2013). Convergence for solutions with finite algebraic moments (which applies to a wider range of initial conditions) has been studied in Cañizo et al. (2017); Murray and Pego (2016b,a).

The approach in Jabin and Niethammer (2003) is based on the entropy-entropy production method and has been recently revisited by the authors of the present paper in Cañizo, Einav, and Lods (2017). It consists in estimating in a careful way the evolution of the *relative free energy*

$$H(\boldsymbol{c}(t)|\boldsymbol{\mathcal{Q}}) := \sum_{i=1}^{\infty} \left(c_i(t) \log \frac{c_i(t)}{\mathcal{Q}_i} - c_i(t) + \mathcal{Q}_i \right), \qquad t \geqslant 0. \tag{1.7}$$

We observe that $H(\boldsymbol{c}(t)|\boldsymbol{\mathcal{Q}})$ is finite whenever the solution $\boldsymbol{c}(t) = (c_i(t))_{i \geqslant 1}$ is nonnegative and has finite density (see for example Cañizo (2007, Lemmas 7.1 and 7.2)). We refer to Cañizo, Einav, and Lods (2017) for more details on the entropy-entropy production method in the context of the Becker-Döring equations.

1.2. **Main results.** A fundamental tool in the application of the entropy method is a *uniform control* of suitable moments of the solution $\boldsymbol{c}(t)$ to (1.1), i.e. the control of suitable weighted-$\ell_1(\mathbb{N})$ estimates. For instance, the analysis of Jabin and Niethammer (2003) deals with subcritical solutions with finite exponential moments and is based on the property that

$$\sum_{i=1}^{\infty} \exp(\eta i)\, c_i(0) < +\infty \implies \sup_{t \geqslant 0} \sum_{i=1}^{\infty} \exp(\eta' i)\, c_i(t) < \infty \tag{1.8}$$

for $\eta > 0$ and some $0 < \eta' < \eta$. This was proved in Jabin and Niethammer (2003), and is to our knowledge the only available result on uniform propagation of moments.

We would like to have a similar information for *algebraic moments* $\sum_{i \geqslant 1} i^k c_i(t)$ or *stretched exponential moments* of the form $\sum_{i \geqslant 1} \exp(\alpha i^{\mu}) c_i$, for some $\alpha > 0$ and $0 < \mu < 1$. Propagation of these moments on a finite time interval is known to hold from the results of Ball et al. (1986) (see Lemma 3.3 hereafter), but the estimate on the time interval $[0, T]$ deteriorates as T increases. We intend to fill this blank with the uniform-in-time propagation results in the next two theorems. For our results we assume that

$$\begin{aligned} &\text{either} \qquad 0 < a_i \leqslant \overline{a}\, i^{\gamma} && \text{for all } i \geqslant 1 \text{ and some } \overline{a} > 0, \quad 0 \leqslant \gamma < 1 \\ &\text{or} \qquad C_1 i \leqslant a_i \leqslant C_2 i && \text{for all } i \geqslant 1 \text{ and some } 0 < C_1 \leqslant C_2. \end{aligned} \tag{1.9}$$

If the second option holds we call $\gamma = 1$ for consistency. We also assume that

$$0 < b_i \leqslant \overline{b}\, a_i, \tag{1.10}$$

for all $i \geqslant 1$ and some $\overline{b} > 0$, and that

$$\lim_{i \to +\infty} \frac{Q_{i+1}}{Q_i} = \frac{1}{z_s} \qquad \text{for some } 0 < z_s < +\infty. \tag{1.11}$$

(Note that z_s is indeed the critical monomer density, in agreement with (1.6).) Additionally we may asume that the critical equilibrium is non-increasing:

$$\text{The sequence } \{Q_i z_s^i\}_i \text{ is non-increasing,} \tag{1.12}$$

though this is not a fundamental requirement and small changes can be made to adapt the proofs if the sequence $\{Q_i z_s^i\}_{i\geqslant i_0}$ is non-increasing only for some fixed $i_0 \in \mathbb{N}$. Regarding the initial data $\boldsymbol{c}^0 = (c_i^0)_{i\geqslant 1}$, we assume it is non-negative and has some finite moments:

$$\sum_{i=1}^{\infty} i^r c_i^0 < +\infty \qquad \text{for } r = \max\{2-\gamma, 1+\gamma\}. \tag{1.13}$$

Theorem 1.1 (***Uniform propagation of moments***). *Assume (1.9)–(1.12), and let $\boldsymbol{c}(t) = (c_i(t))_{i\geqslant 1}$ be a solution to the Becker-Döring equations (1.1) with nonnegative, subcritical initial data $\mathbf{c}(0)$ and density $\varrho < \varrho_s$. Let $k \geqslant \max\{2-\gamma, 1+\gamma\}$ be such that*

$$M_k(0) := \sum_{i=1}^{\infty} i^k c_i(0) < \infty.$$

There exists a constant $C > 0$ depending only on k, $M_k(0)$, the density ϱ and the coefficients $(a_i)_{i\geqslant 1}$, $(b_i)_{i\geqslant 1}$ such that

$$M_k(t) := \sum_{i=1}^{\infty} i^k c_i(t) \leqslant C \qquad \textit{for all } t \geqslant 0.$$

The constant C can be estimated explicitly from the proof. Our second result deals with the uniform propagation of stretched exponential moments in a similar way:

Theorem 1.2 (***Uniform propagation of stretched exponential moments***). *Assume (1.9)–(1.12) hold, with the first option in (1.9) being true (for some $0 \leqslant \gamma < 1$). Let $\mathbf{c}(t) = (c_i(t))_{i\geqslant 1}$ be a solution to the Becker-Döring equations (1.1) with nonnegative, subcritical initial data $\mathbf{c}(0)$ and density ϱ. Let $0 < \mu \leqslant 1-\gamma$ and $\alpha > 0$ be such that*

$$\mathcal{E}_\mu(0) := \sum_{i=1}^{\infty} \exp(\alpha i^\mu) c_i(0) < \infty.$$

There exists a constant $C > 0$ depending only on μ, α, $\mathcal{E}_\mu(0)$, the density ϱ and the coefficients $(a_i)_{i\geqslant 1}$, $(b_i)_{i\geqslant 1}$ such that

$$\mathcal{E}_\mu(t) := \sum_{i=1}^{\infty} \exp(\alpha i^\mu) c_i(t) \leqslant C \qquad \textit{for all } t \geqslant 0.$$

Notice that these two results prove *uniform propagation* of the considered moments whereas the result of Jabin and Niethammer (2003) recalled in (1.8) is of a slightly different nature because of the deterioration of the constant η which measures the strength of the exponential. We do not know whether uniform propagation is true for exponential moments (that is, we do not know whether one can take $\eta' = \eta$ in (1.8)); our method does not immediately apply in this case since the short-time propagation in Lemma 3.3 does not apply to exponential moments.

We mention here that, besides its own interest, Theorem 1.1 plays a crucial role in the determination of the convergence rate to equilibrium for solutions to (1.1) recently established in Cañizo et al. (2017).

1.3. **Method of proof.** A natural attempt to prove the above results would be to directly compute the evolution of $M_k(t)$ or $\mathcal{E}_\mu(t)$. Namely, picking $\phi_i = i^k$ in the weak form (1.3), we get the evolution of $M_k(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t} M_k(t) = \sum_{i=1}^{\infty} \left(a_i c_1(t) c_i(t) - b_{i+1} c_{i+1}(t)\right) \left((i+1)^k - i^k - 1\right),$$

and one may try to obtain a suitable differential inequality for M_k in the spirit of similar results for kinetic equations (see Alonso et al. (2013) for an example on the Boltzmann equation). This method is rather efficient to obtain local-in-time bounds on $M_k(t)$ (or $\mathcal{E}_\mu(t)$) but seems difficult to apply to get uniform bounds on $[0, \infty)$. The difficulty stems from the fact that the "loss term" $b_{i+1}c_{i+1}(t)$ appearing in the evolution does not always compensate the "gain term" $a_i c_1(t) c_i(t)$. A deeper reason for this is that boundedness of moments must depend on the mass of the solution (since moments are never uniformly bounded for supercritical solutions), so any estimate that gives uniform bounds must somehow involve the mass of the solution. In practice, it is the value of $c_1(t)$ that appears when one tries to bound the time evolution of moments, and any uniform estimate seems to require some a priori knowledge on the behaviour of $c_1(t)$. This is in contrast with the situation for the Boltzmann equation (with hard potential interactions) where the optimal Povzner's inequality allows us to control the contribution of gain part of the collision operator by that of its loss part (see for example Bobylev (1996)). It should be remarked that the behaviour of moments for the Boltzmann equation does not depend on the mass of the initial data, but only on which moments are initially finite, which is a fundamental difference with the present case. Another important difference is the fact that there is no creation of moments (of any kind) for the Becker-Döring equations (see Ball et al. (1986)).

We adopt here a different approach relying on a maximum principle. A crucial role in our study will be played by the *tail density* $\mathcal{G}(t) = (G_j(t))_{j \geqslant 1}$ given by

$$G_j(t) = \sum_{i=j}^{\infty} c_i(t), \qquad j \geqslant 1.$$

The main properties of $\mathcal{G}$ which are relevant for us are established in Lemmas 2.2 and 2.4. Tail density was already introduced in Laurençot and Mischler (2002) in order to establish uniqueness of the solution to (1.1) and a variant of it was used in Cañizo (2005) to show strong convergence to equilibrium for a generalised discrete coagulation–fragmentation model.

It is important to notice that moments of $\boldsymbol{c}(t)$ can be estimated by suitable moments of $\mathcal{G}(t)$, so that Theorems 1.1–1.2 can be stated in terms of moments of $\mathcal{G}(t)$ (the rough idea being that the k-th moment of $\boldsymbol{c}$ is equivalent to the $(k-1)$-th moment of $\mathcal{G}$; see Lemma 2.2). Of course, the main interest is that the equation solved by $\mathcal{G}(t)$ is somewhat simpler than (1.1): one has

$$\frac{\mathrm{d}}{\mathrm{d}t} G_j(t) = a_{j-1} c_1(t) \left(G_{j-1}(t) - G_j(t) \right) + b_j \left(G_{j+1}(t) - G_j(t) \right) \qquad j \geqslant 2.$$

The evolution equation for $\mathcal{G}(t)$ depends on $c_1(t)$, and the entire nonlinear structure of the interaction between clusters is driven by it (assuming $c_1(t)$ to be known in (1.1) would yield a linear system of ODEs). Since the coefficient of $c_1(t)$ is nonnegative in the above equation, if one is able to control $c_1(t)$ from above on some given interval, then we can bound the above evolution of $\mathcal{G}(t)$ by a suitable infinite system of differential inequalities, represented by an infinite matrix whose off–diagonal entries are non-negative. This is the key ingredient that yields a maximum principle for the evolution of $\mathcal{G}(t)$ (see Lemmas 2.4 and 2.5). The proof then consists in establishing the existence of suitable supersolutions to the Becker-Döring equations whose moments are strongly related to the moments of $\mathcal{G}(t)$ in order to apply the maximum principle. As already said, this will be possible once a suitable bound on $c_1(t)$ has been established. To prove an a priori bound for $c_1(t)$ we resort to general results of Cañizo (2007) (when $\gamma < 1$) and Cañizo et al. (2017) (when $a_i \sim i$) where the rate of convergence to equilibrium for solutions to (1.1) has been established under mild assumptions on the initial data. Notice that the rate obtained in Cañizo (2007) is far from being optimal but applies to a wide range of initial data, and ensures at least the existence of some explicit time $T > 0$ such that $c_1(t) < z_s$ for $t \geqslant T$. This is enough to apply the method we just described.

1.4. **Organization of the paper.** In the next section we introduce the main tools for the proof of both Theorems 1.1–1.2, namely the introduction of the tail density $\mathcal{G}(t)$ and the maximum principle. The proofs of Theorems 1.1–1.2 are then given in Section 3 after recalling the result on convergence to equilibrium in Cañizo (2007).

2. Tail density and the maximum principle

A key idea for showing our main theorems is to find a quantity, which we will call *the tail density*, that obeys a maximum principle for the equation and whose moments are intimately connected to the moments of $\boldsymbol{c}(t)$, the solution to the Becker-Döring equations.

Definition 2.1. Let $\boldsymbol{c} = \{c_i\}_{i\in\mathbb{N}}$ be a non-negative, summable sequence. We define the *tail density of* $\boldsymbol{c}$ as the sequence $\mathcal{G} = \{G_j\}_{j\in\mathbb{N}}$ given by

$$G_j = \sum_{i=j}^{\infty} c_i, \qquad j \in \mathbb{N}. \tag{2.1}$$

The tail density enjoys the following properties:

Lemma 2.2. *Let* $\boldsymbol{c} = \{c_i\}_{i\in\mathbb{N}}$ *be a non-negative, summable sequence. Then*

(i) the tail density $\mathcal{G} = \{G_j\}_{j\in\mathbb{N}}$ *is a non-negative, non-increasing sequence.*

(ii) For any $k \geqslant 0$

$$\frac{M_{k+1}(\boldsymbol{c})}{k+1} \leqslant M_k(\mathcal{G}) \leqslant M_{k+1}(\boldsymbol{c}). \tag{2.2}$$

(iii) Given $\gamma \in [0,1)$*, let* $\alpha > 0$ *and* $\mu \in (0, 1-\gamma)$*. Introduce*

$$\mathcal{E}_\mu(\boldsymbol{c}) = \sum_{i\geqslant 1} \exp(\alpha\, i^\mu) c_i.$$

Then, there exist $\eta_1, \eta_2 > 0$*, depending only on* μ *and* α*, such that*

$$\eta_1 \sum_{j=1}^{\infty} \psi_j G_j \leqslant \mathcal{E}_\mu(\boldsymbol{c}) \leqslant \eta_2 \sum_{j=1}^{\infty} \psi_j G_j \tag{2.3}$$

where $\psi_j := j^{\mu-1}\exp(\alpha j^\mu)$*, for all* $j \geqslant 1$*.*

Proof. Point (i) is clear from the definition of the tail density. To show (ii) we notice that

$$\frac{i^{k+1}}{k+1} = \int_0^i x^k dx \leqslant \sum_{j=1}^{i} j^k \leqslant i^{k+1}, \qquad \forall i \geqslant 1.$$

Since

$$\sum_{j=1}^{\infty} j^k G_j = \sum_{j=1}^{\infty} j^k \left(\sum_{i=j}^{\infty} c_i\right) = \sum_{i=1}^{\infty} c_i \left(\sum_{j=1}^{i} j^k\right),$$

where we were allowed to change summation due to the non-negativity of the elements, the proof of (ii) complete.

To prove point (iii) we write $c_i = G_i - G_{i+1}$ to obtain

$$\mathcal{E}_\mu(\boldsymbol{c}) = \sum_{i\geqslant 1} \exp(\alpha i^\mu) c_i = \sum_{i\geqslant 1} \exp(\alpha i^\mu)(G_i - G_{i+1})$$
$$= \exp(\alpha) G_1 + \sum_{i\geqslant 2} G_i \left(\exp(\alpha i^\mu) - \exp(\alpha(i-1)^\mu)\right). \quad (2.4)$$

Since

$$\exp(\alpha i^\mu) - \exp(\alpha(i-1)^\mu) \leqslant \alpha\mu(i-1)^{\mu-1}\exp(\alpha i^\mu), \qquad i \geqslant 2$$

we have

$$\begin{aligned}\mathcal{E}_\mu(\boldsymbol{c}) &\leqslant \exp(\alpha)G_1 + \alpha\mu\sum_{i=2}^{\infty}(i-1)^{\mu-1}\exp\left(\alpha i^\mu\right)G_i \\ &\leqslant \exp(\alpha)G_1 + 2^{1-\mu}\alpha\mu\sum_{i=2}^{\infty} i^{\mu-1}\exp\left(\alpha i^\mu\right)G_i \\ &\leqslant \max\left(1,\ 2^{1-\mu}\alpha\mu\right)\sum_{j=1}^{\infty}\psi_j G_j.\end{aligned} \quad (2.5)$$

In addition,

$$\exp(\alpha i^\mu) - \exp(\alpha(i-1)^\mu) \geqslant \alpha\mu i^{\mu-1}\exp(\alpha(i-1)^\mu), \qquad i \geqslant 2.$$

Thus, using the fact that $\exp\left(\alpha j^\mu - \alpha(j-1)^\mu\right) \underset{j\to\infty}{\longrightarrow} 1$ when $0 < \mu < 1$, from (2.4) we conclude that

$$\mathcal{E}_\mu(\boldsymbol{c}) \geqslant \exp(\alpha)G_1 + C\sum_{i=2}^{\infty} i^{\mu-1}\exp\left(\alpha i^\mu\right)G_i \geqslant \max\{1, C\}\sum_{j=1}^{\infty}\psi_j G_j, \quad (2.6)$$

where $C = \alpha\mu\sup_{j\geqslant 2}\exp\left(\alpha(j-1)^\mu - \alpha j^\mu\right)$. This proves the result. □

Remark 2.3. We observe that the equivalence of moments above can be understood by the layercake representation of the weight $\varphi(i) = \exp(\alpha i^\mu)$. If we define a measure on $\mathbb{N}$ by

$$\nu(A) = \sum_{i\in A} c_i, \qquad A \subset \mathbb{N}$$

we find that

$$\mathcal{E}_\mu(\boldsymbol{c}) = \int_{\mathbb{N}} \varphi(x)\nu(\,\mathrm{d}x) = \int_0^\infty \nu\left(\{x\in\mathbb{N}\ :\ \varphi(x) > s\}\right)\mathrm{d}s.$$

We notice that if $s \in [\varphi(j-1), \varphi(j))$ with $j \geqslant 1$ one has that

$$\nu\left(\{x\in\mathbb{N}\ :\ \varphi(x) > s\}\right) = \sum_{i=j}^{\infty} c_i = G_j,$$

while if $0 \leqslant s < 1$, $\varphi(x) > s$ for all $x \in \mathbb{N}$, implying that $\nu\left(\{x\in\mathbb{N}\ :\ \varphi(x) > s\}\right) = G_1$. Thus, we find that

$$\mathcal{E}_\mu(\boldsymbol{c}) = G_1 + \sum_{j=1}^{\infty} G_j \int_{\varphi(j-1)}^{\varphi(j)} \mathrm{d}s = G_1 + \sum_{j=1}^{\infty}\left(\varphi(j)^\alpha - \varphi(j-1)^\alpha\right)G_j. \quad (2.7)$$

We have also the following whenever $\boldsymbol{c}(t)$ is a solution to (1.1):

Lemma 2.4. *Let $\boldsymbol{c}(t)$ be a solution to the Becker-Döring equation with nonnegative, finite-density initial data. Assume (1.9) to hold. Then its associated tail density $\mathcal{G}(t)$ is continuously differentiable, and satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}G_j(t) = a_{j-1}c_1(t)\left(G_{j-1}(t) - G_j(t)\right) + b_j\left(G_{j+1}(t) - G_j(t)\right), \qquad j \geqslant 1. \tag{2.8}$$

In particular, if there exist $t_0 > 0$ and $\omega > 0$ such that

$$c_1(t) \leqslant \omega \qquad \forall t \geqslant t_0,$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}G_j(t) \leqslant a_{j-1}\omega\left(G_{j-1}(t) - G_j(t)\right) + b_j\left(G_{j+1}(t) - G_j(t)\right), \qquad \forall t \geqslant t_0. \tag{2.9}$$

Proof. We notice that for any $k, N \in \mathbb{N}$ with $1 < k \leqslant N$

$$\sum_{i=k}^{N}\left|\frac{\mathrm{d}}{\mathrm{d}t}c_i(t)\right| = \sum_{i=k}^{N}\left|a_ic_1(t)c_i(t) - b_{i+1}c_{i+1}(t) - a_{i-1}c_1(t)c_{i-1}(t) + b_ic_i(t)\right|$$

$$\leqslant 2\overline{a}\varrho\sum_{i=k-1}^{N} i^{\gamma}c_i(t) + 2\overline{b}\sum_{i=k}^{N+1} i^{\gamma}c_i(t) \leqslant C\sum_{i=k-1}^{N+1} i^{\gamma}c_i(t)$$

where we used (1.9). Recalling now that $\sum_{i=1}^{\infty} ic_i(t)$ converges uniformly on any interval thanks to Proposition 3.1 in Ball et al. (1986), we deduce that $\sum_{i=1}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}c_i(t)$ converges uniformly on any interval for any $\gamma \in [0,1]$. Thus, since $\boldsymbol{c}(t)$ is continuously differentiable, so it $\mathcal{G}(t)$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}G_j(t) = \sum_{i=j}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}c_i(t) = \sum_{i=j}^{\infty}\left(W_{i-1}(t) - W_i(t)\right) = W_{j-1}(t)$$

completing the proof. The second assertion follows immediately from (2.8) and the non-negativity of all the elements involved. □

Looking at inequality (2.9) we notice that the infinite system of differential inequalities for tail densities can be represented by an infinite constant matrix with entries only in the diagonal, and above and below it. Moreover, the off–diagonal entries are non-negative. Unsurprisingly, this will entail a *maximum principle* to the system.

For any given vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, we denote by $\boldsymbol{u} \leqslant \boldsymbol{v}$ the case where $u_i \leqslant v_i$ for all $i = 1, \dots, n$. Given $z \in \mathbb{R}$, we also denote $z_+ = \max(z, 0)$ and, if $\boldsymbol{u} = (u^1, \dots, u^n) \in \mathbb{R}^n$, we set $\boldsymbol{u}_+ = (u_+^1, \dots, u_+^n)$.

Lemma 2.5 (***Maximum principle for linear ODE systems***)**.** *Let $T > 0$ and consider the vector of continuously differentiable functions $\boldsymbol{u} = (u_1, \dots, u_n) \colon [0, T) \to \mathbb{R}^n$. Assume that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t) \leqslant A\boldsymbol{u}(t) \qquad \textit{for all } t \in [0, T), \tag{2.10}$$

where A is a constant $n \times n$ matrix whose off-diagonal entries are non-negative. Then, if $\boldsymbol{u}(0) \leqslant 0$ we have that $\boldsymbol{u}(t) \leqslant 0$ for all $t \in [0, T)$.

Proof. Let $0 \leqslant t < T$ be given. Since $\boldsymbol{u}$ is differentiable at t we have that for $0 < s < T - t$

$$\boldsymbol{u}(t+s) \leqslant \boldsymbol{u}(t) + sA\boldsymbol{u}(t) + o(s) = (I + sA)\boldsymbol{u}(t) + o(s).$$

Set $A = (a_{i,j})_{i,j=1,\dots,n}$ and call $s_0 := \inf_{i=1,\dots,n} |a_{i,i}|^{-1} \in (0,+\infty]$. As the off-diagonal entries of A are non-negative we find that for $0 < s < s_* := \min\{s_0, T-t\}$, the matrix $I + sA$ has all entries non-negative. Thus,

$$[\boldsymbol{u}(t+s)]_+ \leqslant [(I+sA)\boldsymbol{u}(t)]_+ + o(s) \leqslant (I+sA)[\boldsymbol{u}(t)]_+ + o(s)$$

for all $0 < s < s_*$. Denoting by $y(t)$ the ℓ_1-norm of $[\boldsymbol{u}(t)]_+$, i.e. $y(t) := \sum_{j=1}^n u_+^j(t)$, we see that

$$y(t+s) \leqslant y(t) + sCy(t) + o(s),$$

where

$$C = \max_{i=1,\dots,n} \sum_{j=1}^n |a_{i,j}| \,.$$

Dividing by s and taking the limit as $s \to 0$ we see that

$$\liminf_{s\to 0^+} \frac{y(t+s) - y(t)}{s} \leqslant Cy(t).$$

A generalised version of Gronwall's lemma, following from a generalised comparison theorem that can be found in Amann (1990, Lemma 16.4, p. 215), implies that

$$y(t) \leqslant y(0) \exp(Ct) \,.$$

Since $y(0) = 0$ we conclude the proof. □

Remark 2.6. The above proof is a simple version of the invariance of the cone of points with non-positive coordinates using the so-called *sub-tangent condition* (as given for example in Amann (1990, Theorem 16.5, p. 215)). A matrix with non-negative off–diagonal entries is known as a *Metzler matrix*, and its sign-preserving properties are well-known. We have given a full proof for the sake of completeness, and since we need the result when we deal with an inequality (and not an equality).

In order to use this maximum principle we define the notion of *supersolution* for the Becker-Döring equations:

Definition 2.7 (***Supersolution***)**.** Let $0 < \varrho < \varrho_s$ and $0 < \omega < z_s$ be given. We say that a non-negative sequence $(r_j)_{j\geqslant 1}$ is a (ω, ϱ)-*supersolution* to Becker-Döring equations if

(1) $r_1 \geqslant \varrho$
(2) For all $j \geqslant 2$ it holds that

$$b_j(r_{j+1} - r_j) + a_{j-1}\omega(r_{j-1} - r_j) \leqslant 0. \tag{2.11}$$

Remark 2.8. Notice that, strictly speaking, a sequence $(r_j)_{j\geqslant 1}$ with the above properties is not a supersolution to (1.1) (in the classical ODEs sense) but rather a supersolution of the system:

$$\frac{\mathrm{d}}{\mathrm{d}t} x_j(t) = a_{j-1}\omega\left(x_{j-1}(t) - x_j(t)\right) + b_j\left(x_{j+1}(t) - x_j(t)\right), \qquad j \geqslant 1 \tag{2.12}$$

with $x_i(t) \leqslant \varrho$ for all $i \geqslant 1$, $t \geqslant 0$. Notice also that, whenever (2.9) holds true, $\mathcal{G}(t)$ is a subsolution of (2.12) on $[t_0, \infty)$.

The values of ω and ϱ are connected to those of $c_1(t)$ and ρ to obtain the following maximum principle:

Proposition 2.9 (***Maximum principle***)**.** *Let $\mathbf{c}(t) = (c_i(t))_{i\geqslant 1}$ be a solution to the Becker-Döring equations with non-negative initial condition $\mathbf{c}(0)$. Assume (1.9)–(1.13) hold and that the density of $\mathbf{c}$ is $0 < \varrho < \varrho_s$. Let $\mathcal{G}(t)$ denote the tail density of $\mathbf{c}(t)$. Take $\omega > 0$ and $0 \leqslant t_0 < t_1$, and denote $I := [t_0, t_1]$. Assume that*

$$c_1(t) \leqslant \omega \qquad \text{for all} \quad t \in I.$$

Let $(r_j)_{j\geqslant 1}$ be a (ω, ϱ)-supersolution to the associated Becker-Döring equations. Then if

$$G_j(t_0) \leqslant r_j \qquad \text{for all } j \geqslant 1,$$

we find that

$$G_j(t) \leqslant r_j \qquad \text{for all } t \in [t_0, t_1] \text{ and all } j \geqslant 1.$$

Proof. Since $[t_0, t_1]$ is compact and the sequence $\mathcal{G}(t) = (G_j(t))_{j\geqslant 1}$ is a non-increasing sequence of continuous functions that converge pointwise to zero, we conclude from Dini's Theorem that

$$\lim_{j\to\infty} \sup_{t\in[t_0,t_1]} G_j(t) = 0.$$

Given $\varepsilon > 0$, set

$$H_j(t) = G_j(t) - r_j - \varepsilon, \qquad \forall t \in [t_0, t_1], \quad j \geqslant 2.$$

There exists an $M \geqslant 1$, independent in t, such that

$$H_{j+1}(t) \leqslant 0 \qquad \forall t \in [t_0, t_1], \quad j \geqslant M. \tag{2.13}$$

In addition, as $H_1(t) = \varrho - r_1 - \varepsilon$, the condition $\varrho \leqslant r_1$ implies that $H_1(t) < 0$. Lemma 2.4 and condition (2.11) for the supersolution sequence imply that

$$\frac{\mathrm{d}}{\mathrm{d}t} H_j(t) \leqslant b_j(H_{j+1}(t) - H_j(t)) + a_{j-1}\omega(H_{j-1}(t) - H_j(t)) \qquad \forall j \geqslant 2 \tag{2.14}$$

on I. Due to (2.13) and the fact that $H_1(t) < 0$ we can consider the system (2.14) for $j = 2, \ldots, M$ only. This system can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} H_2(t) \\ H_3(t) \\ \vdots \\ \vdots \\ H_{M-1}(t) \\ H_M(t) \end{pmatrix} \leqslant \begin{pmatrix} -\alpha_2 & b_2 & 0 & \cdots & 0 \\ a_2\omega & -\alpha_3 & b_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & a_{M-2}\omega & -\alpha_{M-1} & b_{M-1} \\ 0 & \cdots & 0 & a_{M-1}w & -\alpha_M \end{pmatrix} \begin{pmatrix} H_2(t) \\ H_3(t) \\ \vdots \\ \vdots \\ H_{M-1}(t) \\ H_M(t) \end{pmatrix},$$

where $\alpha_j = a_{j-1}\omega + b_j$. As all the off–diagonal entries of the above matrix are non-negative and since our initial conditions imply

$$H_j(t_0) = G_j(t_0) - r_j - \varepsilon < 0,$$

we find that due to our maximum principle (Lemma 2.5)

$$H_j(t) \leqslant 0 \qquad \forall\, 2 \leqslant j \leqslant M, t \in I.$$

Together with the bounds on H_1 and $(H_j)_{j\geqslant M+1}$ we conclude that on I

$$G_j(t) \leqslant r_j + \varepsilon.$$

As ε was arbitrary, we get our desired result. □

As we can see, if ϱ is the density of $\boldsymbol{c}(t)$, the associated (ω,ϱ)-supersolution will control $\mathcal{G}(t)$, with appropriate initial conditions. The question remains as to which ω one may choose. This choice will be crucial to *the existence of a supersolution* that bounds $\mathcal{G}(t)$ at a suitable time. Since in the subcritical case $c_1(t)$ converges to $\overline{z} < z_s$, it seems natural to choose ω close to, but larger than, $\overline{z}$. This is indeed the required ingredient to construct a supersolution. The following lemma, which not only gives us the existence of a supersolution but also gives us moment connections between it and $\mathcal{G}$, is reminiscent to Cañizo (2005, Lemma 3.4).

Lemma 2.10. *Assume* (1.9)–(1.13) *to hold. Let* $0<\varrho<\varrho_s$ *and* $0<\omega<z_s$ *be given. Consider a non-negative, non-increasing sequence* $(g_j)_{j\geqslant 1}$ *that tends to* 0 *as* j *goes to infinity and such that* $g_1 \leqslant \varrho$. *Then, there exists a* (ω,ϱ)*-supersolution* $(r_j)_{j\geqslant 1}$ *to the associated Becker-Döring equations which tends to* 0 *as* j *goes to infinity, and satisfies*

$$g_j \leqslant r_j \qquad \forall j \geqslant 1.$$

Moreover, $(r_j)_{j\geqslant 1}$ *can be chosen so that for any* $1 \leqslant \delta < z_s/\omega$ *and any positive, eventually non-decreasing sequence* $(\phi_j)_{j\geqslant 1}$ *satisfying*

$$\limsup_{j\to+\infty} \frac{\phi_j}{\phi_{j-1}} \leqslant \delta, \tag{2.15}$$

we have that

$$\sum_{j=1}^{\infty} \phi_j r_j \leqslant C\left(1+\sum_{j=1}^{\infty}\phi_j g_j\right) \tag{2.16}$$

where $C>0$ *is a fixed constant that depends only on* $(\phi_j)_{j\geqslant 1}$, ϱ, ω, δ *and the coefficients* $(a_i)_{i\geqslant 1}$, $(b_i)_{i\geqslant 1}$.

Proof. According to (1.5) and (1.11), one notices that $\lim_{j\to\infty} b_j/a_{j-1} = z_s$, we can find $1<\lambda\in(\delta,\frac{z_s}{\omega})$ and $N\geqslant 1$ such that

$$b_j \geqslant \lambda\,\omega\, a_{j-1} \qquad \forall j \geqslant N.$$

For $j \geqslant N$, we set

$$\begin{cases} h_j & := g_j - g_{j+1} \geqslant 0 \\ s_N & := 1 + h_N, \quad s_{j+1} := \max\left\{\dfrac{s_j}{\lambda}, h_{j+1}\right\} \end{cases}$$

and define

$$r_j := \sum_{\ell=j}^{\infty} s_\ell.$$

We will now show that this sequence is well defined and is bounded. Indeed, using the fact that

$$0 \leqslant h_j \leqslant g_j \leqslant g_1 \leqslant \varrho \qquad \forall j \geqslant 1,$$

and the fact that $1 \leqslant s_N \leqslant \varrho+1$ and $1\leqslant\delta<\lambda$, we can use a simple induction to show that

$$0 < s_j \leqslant \varrho + 1 \qquad \forall j \geqslant N.$$

Moreover, as $s_{j+1} \leqslant \frac{s_j}{\lambda} + h_{j+1}$ for all $j \geqslant N$, we have that for any $p \geqslant N$

$$\sum_{j=N}^{p+1} s_j = s_N + \sum_{j=N}^{p} s_{j+1} \leqslant 1 + h_N + \sum_{j=N}^{p} h_{j+1} + \frac{1}{\lambda}\sum_{j=N}^{p} s_j$$
$$\leqslant 1 + g_{N+1} - g_{p+2} + \frac{1}{\lambda}\sum_{j=N}^{p+1} s_j.$$

Due to the non-negativity of g_j and the fact that $1 < \lambda$ we conclude that

$$\sum_{j=N}^{p+1} s_j \leqslant \frac{g_{N+1}+1}{\left(1-\frac{1}{\lambda}\right)}.$$

As p is arbitrary, this shows that the sum converges and thus that $(r_j)_{j\geqslant N}$ is well defined with $\lim_{j\to\infty} r_j = 0$. Moreover, using that $g_{N+1} \leqslant \varrho$ we see from the previous inequality that

$$r_j \leqslant \frac{\lambda(\varrho+1)}{\lambda-1} \qquad j \geqslant N.$$

From its definition, $(r_j)_{j\geqslant N}$ is clearly non-negative and non-increasing. In addition

$$r_j \geqslant \sum_{\ell=j}^{\infty} h_\ell = g_j,$$

where we used the fact that $(g_j)_{j\geqslant 1}$ goes to zero as j goes to infinity. Due to the choice of N,

$$\frac{r_{j-1}-r_j}{r_j - r_{j-1}} = \frac{s_{j-1}}{s_j} \leqslant \lambda \leqslant \frac{b_j}{\omega a_{j-1}}.$$

All of the above show that we have managed to construct a supersolution to the associated Becker-Döring equation from the point $j = N$. We are left with defining it for $j < N$. We set for any $j < N$

$$r_j = \max\{\varrho, r_N\}. \tag{2.17}$$

Clearly, by its definition

$$g_j \leqslant g_1 \leqslant \varrho \leqslant r_j,$$

which also shows that $\varrho \leqslant r_1$. In addition, one checks that

$$a_{j-1}\omega(r_{j-1}-r_j) + b_j(r_{j+1}-r_j) = \begin{cases} 0 & j < N-1 \\ b_{N-1}\left(r_N - \max\{\varrho, r_N\}\right) \leqslant 0 & j = N-1 \end{cases}.$$

Thus (1) and (2) from Definition 2.7, are satisfied up to $j = N-1$. Together with our definition for $j \geqslant N$ we conclude that $(r_j)_{j\geqslant 1}$ is an (ω, ϱ)–supersolution to the associated Becker-Döring equation. Moreover,

$$r_j \leqslant \frac{\lambda(\varrho+1)}{\lambda-1}, \qquad \forall j \geqslant 1.$$

We turn our attention now to the second part of the proof. Due to the conditions on $(\phi_j)_{j\geqslant 1}$ we can find $\delta < \delta_* < \lambda$ and $M \geqslant N \geqslant 1$ such that for all $j \geqslant M$

$$\phi_j \leqslant \phi_{j+1}$$
$$\frac{\phi_j - \phi_{j-1}}{\phi_j} \leqslant 1 - \frac{1}{\delta_*}.$$

Consider the sum $\sum_{j=1}^{\infty} r_j \phi_j$. Using again that $s_{j+1} \leqslant \frac{s_j}{\lambda} + h_{j+1}$ for any $j \geqslant M \geqslant N$, we have that

$$s_j \leqslant r_j = \sum_{\ell=j}^{\infty} s_\ell \leqslant s_j + \frac{1}{\lambda} \sum_{\ell=j}^{\infty} s_\ell + \sum_{\ell=j}^{\infty} h_{\ell+1} \leqslant s_j + \frac{r_j}{\lambda} + g_{j+1}, \qquad j \geqslant M.$$

Thus

$$s_j \leqslant r_j \leqslant \frac{\lambda(s_j + g_{j+1})}{\lambda - 1}.$$

From the above we can estimate that

$$\begin{aligned}
\sum_{j=M+1}^{\infty} \phi_j r_j &\leqslant \frac{\lambda}{\lambda-1} \left(\sum_{j=M+1}^{\infty} \phi_j s_j + \sum_{j=M}^{\infty} \phi_{j-1} g_j \right) \leqslant \frac{\lambda}{\lambda-1} \left(\sum_{j=M+1}^{\infty} \phi_j \left(r_j - r_{j+1} \right) + \sum_{j=M}^{\infty} \phi_j g_j \right) \\
&= \frac{\lambda r_M \phi_M}{\lambda - 1} + \frac{\lambda}{\lambda-1} \left(\sum_{j=M}^{\infty} r_j \left(\phi_j - \phi_{j-1} \right) + \sum_{j=M}^{\infty} \phi_j g_j \right) \\
&\leqslant \frac{\lambda^2 (\varrho+1) \phi_M}{(\lambda-1)^2} + \left(1 + \frac{1}{\lambda-1} \right) \left(1 - \frac{1}{\delta_*} \right) \sum_{j=M}^{\infty} r_j \phi_j + \frac{\lambda}{\lambda-1} \sum_{j=M}^{\infty} \phi_j g_j,
\end{aligned}$$

which implies that

$$\sum_{j=M+1}^{\infty} \phi_j r_j \leqslant \frac{\lambda \delta_*}{\lambda - \delta_*} \left(\frac{\lambda(\varrho+1)\phi_M}{\lambda-1} + \sum_{j=M}^{\infty} \phi_j g_j \right).$$

Thus, as

$$\sum_{j=1}^{M} \phi_j r_j \leqslant \frac{\lambda(\varrho+1)}{\lambda-1} \sum_{j=1}^{M} \phi_j$$

we conclude that

$$\sum_{j=1}^{\infty} \phi_j r_j \leqslant C \left(1 + \sum_{j=1}^{\infty} \phi_j g_j \right)$$

where

$$C = \max \left(\frac{\lambda(\varrho+1)}{\lambda-1} \sum_{j=1}^{M} \phi_j, \frac{\lambda \delta_*}{\lambda - \delta_*} \max \left(1, \frac{\lambda(\varrho+1)\phi_M}{\lambda-1} \right) \right).$$

This completes the proof. □

Remark 2.11. It is important to note that the sequences $(\phi_j)_{j \geqslant 1}$ given by

$$\phi_j = j^k \quad (k \geqslant 0), \qquad \text{or} \qquad \phi_j = \exp\left(j^\mu\right) \quad (0 < \mu < 1), \quad j \geqslant 1$$

both satisfy condition (2.15) with $\delta = 1$. This means that we can build a supersolution with comparable moments, and stretched exponential moments, to those of $\mathcal{G}(t)$. This will be a crucial element in the proof of our main theorems. Note that $\phi_j = \exp(\eta\, j)$ $(j \geqslant 1)$ is also allowed, as long as $\eta < \log\left(\frac{z_s}{\omega}\right)$.

With these tools at hand, we have the main ingredient to prove our main theorem.

3. On the propagation of moments

From the previous section we know that as long as $c_1(t) < z_s$ in a certain time interval, we are able to construct a supersolution to the associated Becker-Döring equations whose moments are strongly related to the moments of $\boldsymbol{c}(t)$, thus allowing us to take advantage of the maximum principle in Lemma 2.9 to obtain a uniform bound. However, the condition on $c_1(t)$ is not necessarily valid at all times. Before we prove our main theorems, we show that one can find an explicit time, $T_0 \geqslant 0$, such that for all $T > T_0$, $c_1(t) < z_s$. Before that time the moments, and stretched exponential moments, grow at most exponentially in time (which was already noted in previous works).

The fact that $c_1(t) < z_s$ after a certain time T_0 is a consequence of a stronger statement about the convergence to equilibrium of the solution to the Becker-Döring equations. The quantitative version we state here uses the relative free energy mentioned in the introduction and can be easily deduced from results in Cañizo (2007) and Cañizo, Einav, and Lods (2017):

Theorem 3.1. *Consider the Becker-Döring equations with coagulation and fragmentation coefficients $(a_i)_{i\geqslant 1}, (b_i)_{i\geqslant 1}$ such that (1.9)–(1.12) hold. Assume that $\mathbf{c}(t) = (c_i(t))_{i\geqslant 1}$ is a solution to the Becker-Döring equations with non-negative, subcritical initial data $\boldsymbol{c}(0)$ satisfying (1.13). Then, there exists a constant $C > 0$, depending only on the coagulation and fragmentation coefficients, the density ϱ, the initial moment of $\boldsymbol{c}$ of order $\max\{2-\gamma, 1+\gamma\}$ and the initial relative free energy, such that*

$$\sum_{i=1}^{\infty} i\,|c_i(t) - \mathcal{Q}_i| \leqslant \frac{C}{\sqrt{1+|\log t|}}, \qquad \forall t > 0. \tag{3.1}$$

Proof. The result in Cañizo (2007) is valid under (1.9)–(1.13) for $\gamma \in [0,1)$ (and holds true for more general discrete coagulation models). For $\gamma = 1$ (i.e., the second option in (1.9)), the rate is actually exponential thanks to a recent result by the authors—see Theorem 1.3 in Cañizo et al. (2017). Notice that for the case $\gamma = 1$ no assumptions on the propagation of moments are needed in Cañizo et al. (2017). □

As a consequence we have:

Corollary 3.2. *Assume the conditions of Theorem 3.1. Then for any $\delta > 0$ one has*

$$c_1(t) < \overline{z} + \delta \qquad \textit{for all } t > T_\delta,$$

where $T_\delta = \max\left(1, \exp\left(\frac{C^2}{\delta^2} - 1\right)\right)$, with $C > 0$ is the explicit constant from Theorem 3.1.

Proof. Since

$$|c_1(t) - \overline{z}| \leqslant \sum_{i=1}^{\infty} i\,|c_i(t) - \mathcal{Q}_i| \leqslant \frac{C}{\sqrt{1+|\log t|}}$$

the result follows immediately. □

The last issue that we need to deal with before being able to tackle our main theorem is the issue of the possible growth of our moments, and stretched exponential moments, in the time until $c_1(t)$ is in the right range to use our machinery from the previous section. This has actually been shown in Ball et al. (1986):

Lemma 3.3 (Ball et al. (1986))**.** *Consider the Becker-Döring equations with coagulation and fragmentation coefficients $(a_i)_{i\geqslant 1}, (b_i)_{i\geqslant 1}$ such that (1.9) holds true. Let $(\phi_i)_{i\geqslant 1}$ be* *a non-negative sequence such that*

$$\begin{cases} \phi_{i+1} - \phi_i \geqslant \varepsilon \phi_1 & \forall i \geqslant 1 \\ \sup_{i\geqslant 1} \dfrac{a_i\left(\phi_{i+1} - \phi_i\right)}{\phi_i} = \mathcal{A}_\phi < \infty, \end{cases} \tag{3.2}$$

for some $\varepsilon > 0$. *Then, if* $\boldsymbol{c}(t)$ *is the solution to the Becker-Döring equations with non-negative initial data* $\mathbf{c}(0)$ *with density* ϱ *and such that*

$$M_\phi(\boldsymbol{c}(0)) := \sum_{i=1}^{\infty} \phi_i c_i(0) < \infty,$$

there exists a positive constant C_ϕ *depending on* $\mathcal{A}_\phi, \varepsilon$ *and* ϱ *such that*

$$M_\phi\left(\boldsymbol{c}(t)\right) := \sum_{i=1}^{\infty} \phi_i c_i(t) \leqslant \exp\left(C_\phi t\right) M_\phi\left(\boldsymbol{c}(0)\right), \qquad \forall t \geqslant 0.$$

Proof. A detailed proof of the result is given in Ball et al. (1986). For the sake of completeness we provide a formal proof here, and a fully rigorous one can be obtained by standard approximation arguments. Using (1.3) with the sequence $(\phi_i)_{i\geqslant 1}$ we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{\infty} \phi_i c_i(t) &= \sum_{i=1}^{\infty} \left(a_i c_1(t) c_i(t) - b_{i+1} c_{i+1}(t)\right)\left(\phi_{i+1} - \phi_i - \phi_1\right) \\ &\leqslant \sum_{i=1}^{\infty} a_i c_1(t) c_i(t)(\phi_{i+1} - \phi_i) + \sum_{i=1}^{\infty} b_{i+1} c_{i+1}(t) \phi_1 \end{aligned}$$

where we used that $(\phi_i)_{i\geqslant 1}$ is non-negative. The first sum is estimated with (3.2):

$$\sum_{i=1}^{\infty} a_i c_1(t) c_i(t)(\phi_{i+1} - \phi_i) \leqslant \mathcal{A}_\phi c_1(t) \sum_{i=1}^{\infty} \phi_i c_i(t) \leqslant \varrho \mathcal{A}_\phi \sum_{i=1}^{\infty} \phi_i c_i(t),$$

while using (3.2) and (1.10) we find that

$$\sum_{i=1}^{\infty} b_{i+1} c_{i+1}(t) \phi_1 \leqslant \phi_1 \bar{b} \sum_{i=2}^{\infty} a_i c_i(t) \leqslant \varepsilon^{-1} \bar{b} \sum_{i=2}^{\infty} a_i c_i(t)(\phi_{i+1} - \phi_i) \leqslant \varepsilon^{-1} \bar{b} \mathcal{A}_\phi \sum_{i=2}^{\infty} \phi_i c_i(t).$$

The result then follows with $C_\phi = \left(\varrho + \varepsilon^{-1}\bar{b}\right)\mathcal{A}_\phi$. ☐

Remark 3.4. The two main types of moments we consider, namely

$$(\phi_j)_{j\geqslant 1} = (j^k)_{j\geqslant 1} \quad (k \geqslant 1), \qquad \text{and} \qquad (\phi_j)_{j\geqslant 1} = (\exp(j^\mu))_{j\geqslant 1} \quad (0 \leqslant \mu \leqslant 1 - \gamma),$$

satisfy the assumptions of Lemma 3.3. On the contrary, the previous lemma cannot be applied to exponential moments (with weight $e^{\mu i}$) if a_i diverges to $+\infty$ with i.

We are now ready to prove our main theorems.

Proof of Theorem 1.1. Since we are dealing with a subcritical solution, using Corollary 3.2 we can find an explicit time $T_0 > 0$ such that

$$c_1(t) < \omega < z_s \qquad \text{for any } t > T_0.$$

Due to Lemma 3.3 we can find an explicit constant $C_k > 0$ such that for all $t \leqslant T_0$

$$M_k(T_0) \leqslant \exp(C_k t) M_k(0).$$

Considering the tail density sequence $\mathcal{G}(t)$, we use Lemma 2.10 to find a supersolution to the associated Becker-Döring equation for $t \geqslant T_0$ such that

$$G_j(T_0) \leqslant r_j \qquad \forall j \geqslant 1$$

and

$$\begin{aligned}\sum_{j=1}^{\infty} j^{k-1} r_j &\leqslant C\left(1+\sum_{j=1}^{\infty} j^{k-1} G_j(T_0)\right) \\ &\leqslant C\left(1+M_k\left(\boldsymbol{c}(t)\right)\right) \leqslant C\left(1+M_k(0)\exp(CT_0)\right),\end{aligned} \tag{3.3}$$

where we have used Lemma 2.2 and 3.3. According to the maximum principle, Proposition 2.9, and the fact that $c_1(t) < \omega$ for $t > T_0$ we find that

$$G_j(t) \leqslant r_j \qquad \text{for all } t \geqslant T_0, \text{ for all } j \geqslant 1,$$

and thus, using Lemma 2.2 again, and (3.3), we have that for all $t \geqslant T_0$

$$M_k(t) \leqslant (k+1)\sum_{j=1}^{\infty} j^{k-1} G_j(t) \leqslant (k+1)\sum_{j=1}^{\infty} j^{k-1} r_j \leqslant C\left(1+M_k(0)\exp(CT_0)\right).$$

This concludes he proof. □

Proof of Theorem 1.2. We set

$$\mathcal{E}_\mu(t) = \sum_{i=1}^{\infty} \exp\left(\alpha\, i^\mu\right) c_i(t), \qquad t \geqslant 0.$$

The link between $\mathcal{E}_\mu(t)$ and the tail density $\mathcal{G}(t)$ is given by (2.3), namely

$$\eta_1 \sum_{j=1}^{\infty} \psi_j G_j(t) \leqslant \mathcal{E}_\mu(t) \leqslant \eta_2 \sum_{j=1}^{\infty} \psi_j G_j(t) \qquad \forall t \geqslant 0 \tag{3.4}$$

for some positive constants $\eta_1, \eta_2 > 0$ depending only on α, μ and $\psi_j := \alpha\mu j^{\mu-1}\exp(\alpha j^\mu)$, for all $j \geqslant 1$. At this point we just mimic the proof of Theorem 1.1. Namely, we find an explicit time $T_0 > 0$ such that for any $t > T_0$ $c_1(t) < \omega < z_s$. Then, using Lemma 2.10, and noting that our ψ_j satisfies its conditions, we find a supersolution to the associated Becker-Döring equation, $(r_j)_{j\geqslant 1}$ such that $G_j(T_0) \leqslant r_j$ $(j \geqslant 1)$ and

$$\sum_{j=1}^{\infty} \psi_j r_j \leqslant C(1+\sum_{j=1}^{\infty} \psi_j G_j(T_0)) \leqslant C(1+\eta_1^{-1}\mathcal{E}_\mu(T_0))$$

according to (3.4). Invoking now Proposition 2.9, we get $G_j(t) \leqslant r_j$ for all $t \geqslant T_0$ and all $j \geqslant 1$. Using again (3.4), we have then

$$\mathcal{E}_\mu(t) \leqslant \eta_2 \sum_{j=1}^{\infty} \psi_j G_j(t) \leqslant \eta_2 \sum_{j=1}^{\infty} \psi_j r_j \leqslant C\,\eta_2(1+\eta_1^{-1}\mathcal{E}_\mu(T_0)), \quad \forall t \geqslant T_0.$$

The proof is now complete by using Lemma 3.3 with $\phi_i = \exp\left(\alpha i^\mu\right)$ $(i \geqslant 1)$. □

Acknowledgements

JAC was supported by project MTM2014-52056-P, funded by the Spanish government and the European Regional Development Fund. AE was partially supported by EPSRC grant EP/L002302/1 and partially supported by the Austrian Science Fund (FWF) grant M 2104-N32. We would like to thank André Schlichting for useful comments on our previous work, which led to us realising that no uniform bounds for moments were available.

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