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# A unisex stochastic mortality model to comply with EU Gender Directive\*

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## Abstract

EU Gender Directive ruled out discrimination against gender in charging premium for insurance products. This prohibition prevents the use of the standard actuarial fairness principle to price life insurance products, with an evident negative effect on pricing efficiency. According to current actuarial practice, unisex premiums are calculated with a simple weighting rule of the gender-specific life tables. Up to our knowledge, there seems to be neither unisex fairness principle in the actuarial literature, nor unisex mortality model. This paper is the first attempt to fill this gap. First, we introduce a unisex fairness principle and the corresponding unisex fair premium. Then, we provide a unisex stochastic mortality model for the mortality intensity that is underlying the pricing of a life portfolio of females and males belonging to the same cohort. Finally, we calibrate the unisex mortality model using the unisex fairness principle. We find that the weighting coefficient between the males' and females' own mortalities depends mainly on the quote of portfolio relative to each gender, on the age, and on the type of insurance product. We also investigate the impact of the correlation among the two mortality intensities on the weighting coefficient. The knowledge and the adoption of

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a proper unisex mortality model should help life insurance companies in many tasks, including pricing, reserving, profit testing, calculation of solvency capital requirements and, ultimately, should result in improved competitiveness.

**Keywords.** Actuarial fairness, unisex tariff, stochastic mortality intensity, Gender Directive, life table, doubly stochastic process.

**JEL classification:** C1, C13, C18, C38, J11.

## 1 Introduction and motivation

On 1 March 2011 the European Court of Justice ruled that differences in insurance premiums based on the policyholder's gender are discriminatory (European Union Directive 2004/113/EC, also called Gender Directive), and that gender equality in the European Union must be ensured from 21 December 2012. Therefore, insurance policies issued from 21 December 2012 cannot be priced according to the insured's gender. Life insurance policies pay benefits in case of occurrence of an event related to the demographic status of the insured, such as death or survivorship at a certain date or within a given time frame. According to the actuarial equivalence principle, the premium of an insurance policy is based on the likelihood that the event will occur, i.e., it depends on the probability that the insured will die or survive within a given time frame. It is well known that the death and survival probabilities of an individual are strongly related to his/her gender: women are more likely than men to survive at every given future date, everything else (age, health status etc.) being equal. The use of gender-specific mortality tables permits the calculation of actuarially fair premiums of insurance policies. The prohibition of using gender-specific mortality tables since December 2012 increases the difficulty in respecting actuarial fairness, and the risk of insolvency.

The implications of the Gender Directive have been enormous, both at economic level (on the insurance market) and at actuarial-financial level. For the economic aspect, for instance Sass and Seifried (2014) [11] analyze the effects of mandatory unisex tariffs on the optimal insurance demand and point out that it has an adverse effect on the insurer's portfolio.

They show that unisex tariffs might cause market distortion, hence reduce the overall social welfare. Schmeiser et. al. (2014) [12] discuss unisex insurance pricing from the insurance industry's, the regulator's and ethical point of view. More importantly, through an international consumer survey conducted in the UK, Germany, France, Italy and Switzerland, they assess the customer's acceptance of price differentiation for diverse insurance products. Thiery and Van Schoubroeck (2006) [14] discuss also the fairness and equality in actuarial risk selection, but from a legal standpoint. They clarify the conditions for reaching a fair insurance-differentiation scheme. For the actuarial aspect, Guillén (2012) [5] suggests that gender information shall be taken into consideration when analyzing the insurance companies' data, despite of the ban on the gender discrimination. In the future, other indicators like lifestyle information can serve as a better predictor than gender in the actuarial modelling, if insurance companies can pay more attention to data quality rather than data quantity. Ornelas and Guillén (2013) [6] model the general population in Mexico and compare it with Mexican unisex life tables that are used for insurance purposes. They suggest that unisex tariffs should be based on reasonable assumptions about the proportions of males and females in the mixed portfolio. They discuss the possible bias resulting from using the unisex life tables, particularly when the proportion of the male and female policyholders is not balanced.

In this work we do not discuss the economic implications of the Gender Directive, but focus only on the actuarial aspects. An important issue is: What is the best way to reach actuarial fairness and respect the Gender Directive? The actuarial literature on this topic is silent. The current practice in insurance companies seems to be the calculation of a unisex tariff obtained by mixing the life tables of the two sexes, with a correction that takes into account the insurance product considered and gives more weight to the life table of the high-risk gender. With this procedure, the usual actuarial fairness principle is not respected because the total premiums collected on a mixed life portfolio (i.e., a portfolio of life insurance products written on both males and females) are not equal to the total premiums charged to the same portfolio before the Gender Directive. The lack of actuarial fairness reflects in difficult reserving procedures and, more in general, in lack of competitiveness.

This paper aims to address this relevant topic by making the following two main con-

tributions. *First*, we introduce a *unisex fairness principle*, that should be used to price unisex tariffs for mixed portfolios; at the same time, we define the corresponding *unisex fair premium*, that respects the unisex fairness principle. *Second*, we push this process one step further and suggest what should be the underlying *unisex mortality model* implicitly used to calculate the unisex fair premium: we start by proposing the unisex mortality intensity as a weighted average of the underlying gender-specific mortality intensities, and then we fix the weights in such a way that the actuarial fairness principle is satisfied. Following a well established stream of actuarial literature on the doubly stochastic setup for the stochastic mortality (see [10], [2], [1], [13], [9]), we model the stochastic mortality intensity (i.e., the stochastic force of mortality) of the single gender as an affine process. In particular, we model it as an Ornstein-Uhlenbeck non-mean reverting affine process that is a natural extension of the Gompertz law for the force of mortality. As a consequence, the unisex mortality intensity is a two-factor continuous-time cohort-based mortality model and has the desirable feature of providing closed-form survival probabilities. Finally, we exploit the closed-form survival functions and calibrate the unisex mortality model in practical situations, considering a variety of cohorts and insurance products. The calibrations show that the weights identifying the unisex mortality intensity depend on many factors: mainly (and expectedly), the quote of portfolio relative to each gender, but also the age, the cohort, the type of insurance product, the duration of the policy and the correlation coefficient between the male's and the female's mortalities.

Risk management of any kind requires the knowledge of the underlying risk process, and the asset-liability management of a mixed life portfolio makes no exception. The knowledge of the unisex mortality process underlying the mixed portfolio should help insurers in many tasks, including (i) forecasts of future cash flows of the portfolio, (ii) calculation of the variance and higher moments of the present value of future liabilities, (iii) calculation of the safety loading, (iv) calculation of the stochastic mathematical reserve, (v) performance of profit testing techniques (vi) calculation of Solvency Capital Requirements etc. Our unisex mortality model provides a picture of the dynamics of the mixed portfolio that insurers have to price, and should facilitate accurate pricing and reserving, improving the companies' competitiveness.

The remaining of the paper is as follows. Section 2 describes the current practice used to price unisex tariffs. Section 3 formulates the concepts of unisex fairness principle and unisex fair premium. Section 4 models the unisex mortality intensity as a two-factor stochastic cohort-based continuous-time intensity process, deriving closed-form expressions for the unisex survival function. Section 5 introduces the concept of fair unisex mortality intensity and fair weighting parameter, and computes them in a variety of life insurance products. Section 6 reports the numerical part: (i) we calibrate the gender-based mortality intensity processes for a number of UK cohorts, (ii) we find the fair weighting coefficient, (iii) we compare the fair weighting coefficient with the volume-related weight. We do this first with fixed correlation between the males' and females' mortality intensities, and then with the estimated sample correlation. Section 7 concludes.

## 2 Unisex premium: current practice

It is rather difficult to report on the current actuarial practice for unisex pricing. The main reason is the fact that, to our knowledge, after the Gender Directive the insurance companies have not made clear and detailed disclosure of their pricing process. According to the Italian Guidelines on the application of the Gender Directive to life insurance products ([4]), regarding the unisex tariffs we read:

*...the unisex demographic basis ...can be defined on the basis of*  
*- prevailing risk or*  
*- weighted risk, assuming a prudent mix of insured of both genders that represent an estimate of the theoretical insured population...*

If the insurer takes the view only of the *prevailing risk*, then the premium of the pure endowment will be based on the females' mortality, and the premium of the term insurance will be based on the males' mortality. This pricing procedure is certainly safe and prudent, but gives raise to severe competitiveness and adverse selection issues. Therefore, we consider it inappropriate.

Insurers seem to incorporate both principles of prevailing risk and weighted risk in pricing unisex tariffs. The unisex tariff is currently based on a simple mix of the gender-based life tables, that takes into account the risk covered and gives more weight to the prevailing risk. Therefore, the unisex life table used to price a pure endowment gives more weight to the females' life table, and the unisex life table used to price a term insurance gives more weight to the males' life table.

It is easy to show that this mixing procedure fails to respect actuarial fairness at the portfolio level. Indeed, using this mixing procedure the total unisex premiums collected on the mixed portfolio are not equal to the total fair premiums that were collected on the two gender-specific subportfolios before the Gender Directive, when the respect of actuarial fairness was not an issue. The systematic violation of actuarial fairness results in difficulties in the reserving process and in reduced competitiveness. In this paper, we take the view of the weighted risk, but aim primarily to the achievement of actuarial fairness at the portfolio level.

### 3 Unisex fairness principle and unisex fair premium

In this section we show how the unisex pricing for a mixed portfolio should be done in order to reach actuarial fairness at the global portfolio level.

Consider a life insurance company operating on a time horizon  $[0, T]$ ,  $T < \infty$ . The company manages a portfolio of identical life insurance policies (e.g., pure endowment with duration  $T$ , term insurance with duration  $T$ ) issued to two groups of policyholders: one cohort with  $m$  homogenous male policyholders and one cohort with  $n$  homogenous female policyholders. All the policyholders are aged  $x$  at time 0. We will call such a portfolio a “mixed portfolio”, because it contains both males and females policyholders. For simplicity, let us assume that the cohort mortality tables of the males and females of generation  $x$  are given, respectively, by the vectors

$$[p_x^m, p_{x+1}^m, \dots, p_{\omega-1}^m] \tag{1}$$

and

$$[p_x^f, p_{x+1}^f, \dots, p_{\omega-1}^f], \quad (2)$$

where  $p_y^i$  is the one-year survival probability of the individual of gender  $i = m, f$  at age  $y \in \{x, x + 1, \dots, \omega - 1\}$ , and  $\omega$  is the extreme age (e.g.  $\omega = 120$ ). Assume that the fair premium for the male of the policy under consideration, using vector (1), is  $P_m$ , and the fair premium for the female of the policy under consideration, using vector (2), is  $P_f$ .

Disregarding safety loadings and commissions, the total fair premiums collected before December 2012 for the mixed portfolio under consideration was

$$m \cdot P_m + n \cdot P_f,$$

because before the Gender Directive the mixed portfolio consisted in two subportfolios, one with  $m$  males and fair premium  $P_m$ , and the other with  $n$  females and fair premium  $P_f$ . Since December 2012, the most natural way to reach the actuarial fairness at the portfolio level is to charge the same total premiums. Therefore, the unisex premium  $P_u$  charged by the insurance company to each of the  $m + n$  policyholders of the mixed portfolio must satisfy

$$(m + n) \cdot P_u = m \cdot P_m + n \cdot P_f.$$

Therefore, we can define the following unisex fairness principle:

**Definition 3.1** (Unisex fairness principle and unisex fair premium). *For a given portfolio of  $m$  male policyholders and  $n$  female policyholders, whose fair premiums are  $P_m$  and  $P_f$  respectively, we say that the unisex tariff  $P_u$  is calculated according to the unisex fairness principle if*

$$P_u = w \cdot P_m + (1 - w) \cdot P_f, \quad (3)$$

where

$$w = \frac{m}{m + n}. \quad (4)$$

*In this case, the unisex tariff  $P_u$  is called unisex fair premium.*

From this definition, it is evident that the actuarial fairness at portfolio level can be

reached only by charging the unisex fair premium.

It can be interesting and useful to interpret the unisex fair premium  $P_u$  as the premium of a representative unisex policyholder of the mixed portfolio, and to analyze the relationship between the survival probabilities of the unisex policyholder and those of males and females. The relationship depends on the type of policy and on its duration. As an illustration, we consider two classical life insurance products, that are the building bricks for every life insurance product: (1) pure endowment; (2) term insurance.

### Pure endowment

For the male, the gender-based fair premium of a pure endowment insurance contract which pays out a unitary payment if the contract holder survives the maturity date  $T$  is

$$P_m = B(0, T) \cdot {}_T p_x^m$$

while that for the female is

$$P_f = B(0, T) \cdot {}_T p_x^f,$$

where  $B(0, T)$  is the financial discount factor from  $T$  to 0 and  ${}_T p_x^i$ ,  $i = m, f$ , is the  $T$ -years survival probability for individual of gender  $i$  aged now  $x$ . The unisex fair premium for the pure endowment is

$$P_u = w \cdot P_m + (1 - w) \cdot P_f = B(0, T) \cdot {}_T p_x^u$$

where

$${}_T p_x^u = w \cdot {}_T p_x^m + (1 - w) \cdot {}_T p_x^f, \quad (5)$$

Therefore, managing a mixed portfolio of pure endowments of duration  $T$  written on  $m$  males and  $n$  females is equivalent to managing a portfolio of  $m + n$  pure endowments issued to unisex policyholders whose unisex  $T$ -years survival probability  ${}_T p_x^u$ , given by (5), is a weighted average of the  $T$ -years survival probabilities of males and females of the portfolio, with weights  $w$  and  $(1 - w)$ , that are volume-related and given by (4).

It is easy to show that this pricing procedure does not correspond to mixing the life tables of males and females with volume-related weights. Indeed, simply mixing the life tables with

volume-related weights produces (1-year) survival rates that are mixed with volume-related weights, but, for  $T \geq 2$  produces  $T$ -years survival probabilities that are not weighted averages of the gender-based  $T$ -years survival probabilities with volume-related weights.<sup>1</sup>

### Term insurance

For the male, the gender-based fair premium of a term insurance contract which pays out a unitary death benefit at the end of year of death if the contract holder does not survive the maturity date  $T$  is

$$P_m = \sum_{k=0}^{T-1} B(0, k+1) \cdot {}_{k/1}q_x^m$$

while that for the female is

$$P_f = \sum_{k=0}^{T-1} B(0, k+1) \cdot {}_{k/1}q_x^f,$$

where  ${}_{k/1}q_x^i$ ,  $i = m, f$ , is the  $k$ -year deferred death probability, i.e. the probability of dying in year  $(k, k+1]$  for individual of gender  $i$  aged now  $x$ . The unisex fair premium for the term insurance is

$$P_u = w \cdot P_m + (1-w) \cdot P_f = \sum_{k=0}^{T-1} B(0, k+1) \cdot {}_{k/1}q_x^u$$

where

$${}_{k/1}q_x^u = w \cdot {}_{k/1}q_x^m + (1-w) \cdot {}_{k/1}q_x^f \quad \text{for all } k = 0, 1, \dots, T-1. \quad (6)$$

Therefore, managing a portfolio of term insurances of duration  $T$  written on  $m$  males and  $n$  females is equivalent to managing a portfolio of  $m+n$  term insurances issued to unisex policyholders whose unisex  $k$ -year deferred death probability  ${}_{k/1}q_x^u$ , given by (6), is a weighted average of the  $k$ -year deferred death probabilities of males and females of the portfolio, for every  $k = 0, 1, \dots, T-1$ , with weights  $w$  and  $(1-w)$ , that are volume-related and given by (4).

As before, it is easy to show that this pricing procedure does not correspond to mixing the

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<sup>1</sup>This is due to the fact that the  $T$ -years survival probability for age  $x$  is the product of the  $T$  1-year survival rates at ages  $x, x+1, \dots, x+T-1$ .

life tables of males and females with volume-related weights.

## 4 Unisex stochastic mortality model

In the previous section, we have seen that the achievement of actuarial fairness boils down to a simple volume-related mixing of survival or deferred death probabilities, depending on the nature of the underlying contract. It is unclear whether this corresponds precisely to a volume-related mixing of the two underlying gender-specific mortality models. Generally, this is not the case. In this section, we propose a two-factor stochastic continuous-time unisex mortality model that is a weighted average of the two underlying gender-specific stochastic mortality models. In Section 5 we will show how to select the weights in order to have a unisex mortality model that produces exactly the unisex fair premium calculated according to the unisex fairness principle of Definition 3.1.

Let us introduce a complete filtered probability space  $(\Omega, \mathfrak{F}, \mathbb{P})$  and a filtration  $\mathcal{F}_t$  of sub- $\sigma$ -algebras representing the state of information at time  $0 \leq t \leq T$ . As in Section 2, consider an insurance company that manages a mixed portfolio with  $m$  male policyholders and  $n$  females policyholders, who have the same age  $x$  at time 0. For notational convenience, and because we are assuming that males and females of the mixed portfolio belong to the same cohort and have the same initial age  $x$ , throughout this section we will omit the dependence on  $x$  of the mortality processes. The stochastic mortality intensities of the two groups are described by two different Ornstein-Uhlenbeck processes (OU processes) with positive drift and no mean reversion:<sup>2</sup>

$$\begin{aligned} d\lambda^m(t) &= \mu_m \lambda^m(t) dt + \sigma_m dW^m(t), \\ d\lambda^f(t) &= \mu_f \lambda^f(t) dt + \sigma_f dW^f(t), \end{aligned} \tag{7}$$

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<sup>2</sup>To keep the treatment as simple as possible, in this paper we do not present all the mathematical arsenal that is behind the introduction of affine processes for the stochastic mortality intensity, i.e. the doubly stochastic setup for the stochastic mortality. To be precise, in this framework the time of death is modelled as the first jump time of a doubly stochastic processes with intensity  $\lambda$ . A complete exposition is beyond the scope of this paper, and we refer the interested reader to the papers on the doubly stochastic setup cited in Section 1.

where  $\mu_i > 0$  and  $\sigma_i > 0$  for  $i = m, f$ , and  $W_m$  and  $W_f$  are two standard Brownian motions under the real world measure  $\mathbb{P}$ , correlated with a correlation coefficient  $\rho$ . So, there is a Brownian motion  $W^{\perp m}$ , independent of  $W^m$  such that  $W^f = \rho W^m + \sqrt{1 - \rho^2} W^{\perp m}$ . The OU process for the mortality intensity is a natural stochastic generalization of the Gompertz law for the force of mortality and is introduced by [9], where its properties and the conditions for its biological reasonableness have been discussed.

Standard properties of affine processes allow us to write the survival probability of a male and a female policyholder in closed-form:

$$S^i(t, T) = E \left[ \exp \left\{ - \int_t^T \lambda^i(u) du \right\} \middle| \mathcal{F}_t \right] = \exp \{ \alpha_i(T - t) + \beta_i(T - t) \lambda^i(t) \}, \quad i = f, m \quad (8)$$

$$\alpha_i(t) = \frac{\sigma_i^2}{2\mu_i^2} t - \frac{\sigma_i^2}{\mu_i^3} e^{\mu_i t} + \frac{\sigma_i^2}{4\mu_i^3} e^{2\mu_i t} + \frac{3\sigma_i^2}{4\mu_i^3}, \quad i = f, m$$

$$\beta_i(t) = \frac{1}{\mu_i} (1 - e^{\mu_i t}), \quad i = f, m$$

The key idea of this paper is to model the mortality intensity of a representative unisex policyholder of the mixed portfolio as a weighted average of the males' and females' mortality intensities, according to the following definition:

**Definition 4.1** (Unisex mortality intensity). *For a mixed portfolio of male and female policyholders, whose stochastic mortality intensities are  $\lambda^m$  and  $\lambda^f$  respectively, we define the  $\xi$ -driven unisex mortality intensity by mixing the male and female intensities with the weight  $\xi \in [0, 1]$ :*

$$\lambda_\xi^u(t) = \xi \lambda^m(t) + (1 - \xi) \lambda^f(t). \quad (9)$$

Clearly, Definition 4.1 defines a *family* of unisex mortality intensities, the members of which are identified by the weight  $\xi \in [0, 1]$ . The correct identification of the weight  $\xi$  is the subject of the next section.

From (7) and (9), we obtain

$$d\lambda_\xi^u(t) = \left( \xi \mu_m \lambda^m(t) + (1 - \xi) \mu_f \lambda^f(t) \right) dt + \xi \sigma_m dW^m(t) + (1 - \xi) \sigma_f dW^f(t). \quad (10)$$

Note that as  $\mu_m$  is usually not identical to  $\mu_f$  and  $\lambda_\xi^u$  does not follow a simple OU process

as  $\lambda^m$  and  $\lambda^f$ . Despite this, the survival probability related to the mixed mortality intensity  $\lambda_\xi^u$  can still be computed nicely.<sup>3</sup>

**Lemma 4.2.** *Conditional on  $t$ , the survival probability for the remaining time  $\tau = T - t$  related to the mixed mortality intensity  $\lambda_\xi^u$  in (9) is given by*

$$S_\xi^u(t, T) = E \left[ \exp \left\{ - \int_t^T \lambda_\xi^u(s) ds \middle| \mathcal{F}_t \right\} \right] = \exp \left\{ \alpha_u(\tau) + \beta_{1,u}(\tau) \lambda^m(t) + \beta_{2,u}(\tau) \lambda^f(t) \right\}, \quad (11)$$

with

$$\begin{aligned} \beta_{1,u}(\tau) &= \frac{\xi}{\mu_m} (1 - e^{\mu_m \tau}), \\ \beta_{2,u}(\tau) &= \frac{1 - \xi}{\mu_f} (1 - e^{\mu_f \tau}), \end{aligned}$$

and

$$\begin{aligned} \alpha_u(\tau) &= \frac{1}{4\mu_f^3 \mu_m^3 (\mu_f + \mu_m)} \left\{ (\mu_f + \mu_m) \left[ \mu_m^3 \sigma_f^2 (3 - 4e^{\mu_f \tau} + e^{2\mu_f \tau} + 2\mu_f \tau) (\xi - 1)^2 \right. \right. \\ &\quad \left. \left. + (3 - 4e^{\mu_m \tau} + e^{2\mu_m \tau}) \mu_f^3 \sigma_m^2 \xi^2 + 2\mu_f^3 \mu_m \sigma_m^2 \xi^2 \tau \right] - 4\mu_f \mu_m \rho \sigma_f \sigma_m \xi (\xi - 1) \right. \\ &\quad \left. \cdot \left[ (1 - e^{\mu_f \tau}) \mu_m^2 + \mu_f^2 (1 - e^{\mu_m \tau} + \mu_m \tau) + \mu_f \mu_m (1 - e^{\mu_f \tau} - e^{\mu_m \tau} + e^{(\mu_f + \mu_m) \tau} + \mu_m \tau) \right] \right\}. \quad (12) \end{aligned}$$

*Proof.* Following Duffie et al. (2000) [3], and the fact that  $\lambda^u$  is a linear function of the two state variables  $\lambda^m$  and  $\lambda^f$ , the survival probability related to the mixed mortality intensity  $\lambda^u$  can be expressed as follows:

$$E \left[ \exp \left\{ - \int_t^T \lambda_\xi^u(s) ds \right\} \middle| \mathcal{F}_t \right] = \exp \left\{ \alpha_u(t) + \beta_{1,u}(t) \lambda^m(t) + \beta_{2,u}(t) \lambda^f(t) \right\}.$$

where  $\alpha_u(t)$ ,  $\beta_{1,u}(t)$  and  $\beta_{2,u}(t)$  follow some ODEs. In order to find out the ODEs, we need to discuss the following conditions.

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<sup>3</sup>In the following, in order to simplify notation, we will omit the subscript  $\xi$  in the functions  $\alpha$ ,  $\beta_1$  and  $\beta_2$

- The mean of the state variables shall satisfy the affine structure:

$$\begin{pmatrix} \mu_m \lambda^m \\ \mu_f \lambda^f \end{pmatrix} := K_0 + K_1 \begin{pmatrix} \lambda^m \\ \lambda^f \end{pmatrix} \Rightarrow K_0 = 0, K_1 = \begin{pmatrix} \mu_m & 0 \\ 0 & \mu_f \end{pmatrix}$$

- The volatility of the state variables shall satisfy the affine structure:

$$\begin{pmatrix} \sigma_m & 0 \\ \rho\sigma_f & \sqrt{1-\rho^2}\sigma_f \end{pmatrix} \begin{pmatrix} \sigma_m & \rho\sigma_f \\ 0 & \sqrt{1-\rho^2}\sigma_f \end{pmatrix} = \begin{pmatrix} \sigma_m^2 & \rho\sigma_f\sigma_m \\ \rho\sigma_f\sigma_m & \sigma_f^2 \end{pmatrix} \\ := H_0 + H_1 \begin{pmatrix} \lambda^m \\ \lambda^f \end{pmatrix} \Rightarrow H_0 = \begin{pmatrix} \sigma_m^2 & \rho\sigma_f\sigma_m \\ \rho\sigma_f\sigma_m & \sigma_f^2 \end{pmatrix}, H_1 = 0.$$

- $\lambda^u$  is a linear function of  $\lambda^m$  and  $\lambda^f$ :

$$(\xi \quad 1 - \xi) \begin{pmatrix} \lambda^m \\ \lambda^f \end{pmatrix} := \rho_0 + \rho_1 \begin{pmatrix} \lambda^m \\ \lambda^f \end{pmatrix} \Rightarrow \rho_0 = 0, \rho_1 = (\xi \quad 1 - \xi)$$

After having determined all the necessary parameters, we can write down the conditions for  $\alpha_u(t)$ ,  $\beta_{1,u}(t)$  and  $\beta_{2,u}(t)$ :

$$\begin{aligned} \beta'_{1,u}(t) &= \xi - \mu_m \beta_{1,u}(t), \text{ with } \beta_{1,u}(T) = 0 \\ \beta'_{2,u}(t) &= (1 - \xi) - \mu_f \beta_{2,u}(t), \text{ with } \beta_{2,u}(T) = 0 \\ \alpha'_u(t) &= -\frac{1}{2}(\beta_{1,u}(t) \quad \beta_{2,u}(t)) H_0 \begin{pmatrix} \beta_{1,u}(t) \\ \beta_{2,u}(t) \end{pmatrix} \\ &= -\frac{1}{2}(\beta_{1,u}^2(t)\sigma_m^2 + \beta_{2,u}^2(t)\sigma_f^2 + 2\beta_{1,u}\beta_{2,u}\rho\sigma_f\sigma_m), \text{ with } \alpha_u(T) = 0. \end{aligned}$$

When we consider survival probabilities, we do not have terminal conditions,  $\alpha_u(T) = \beta_{1,u}(T) = \beta_{2,u}(T) = 0$ . Instead it holds  $\alpha_u(0) = \beta_{1,u}(0) = \beta_{2,u}(0) = 0$ . Noticing the fact that  $\beta'_i(\tau) = -\beta'_i(t)$ ,  $i = 1, 2$ , and  $\alpha'_u(\tau) = -\alpha'_u(t)$ , the above three ODEs can be

transformed to

$$\begin{aligned}\beta'_{1,u}(\tau) &= \mu_m \beta_{1,u}(\tau) - \xi, \text{ with } \beta_{1,u}(0) = 0 \\ \beta'_{2,u}(\tau) &= \mu_f \beta_{2,u}(\tau) - (1 - \xi), \text{ with } \beta_{2,u}(0) = 0 \\ \alpha'_u(\tau) &= \frac{1}{2} (\beta_{1,u}^2(\tau) \sigma_m^2 + \beta_{2,u}^2(\tau) \sigma_f^2 + 2\beta_{1,u}\beta_{2,u}\rho\sigma_f\sigma_m), \text{ with } \alpha_u(0) = 0.\end{aligned}$$

with  $\tau := T - t$ . Solving the ODEs lead to the expressions for  $\beta_{1,u}(\tau)$ ,  $\beta_{2,u}(\tau)$  and  $\alpha_u(\tau)$ .  $\square$

Figure 1 plots the survival functions for males and females, and the joint one for UK population, cohort 1950, initial age 35, with  $\xi = 0.5$  and  $\rho = 0.95$ .

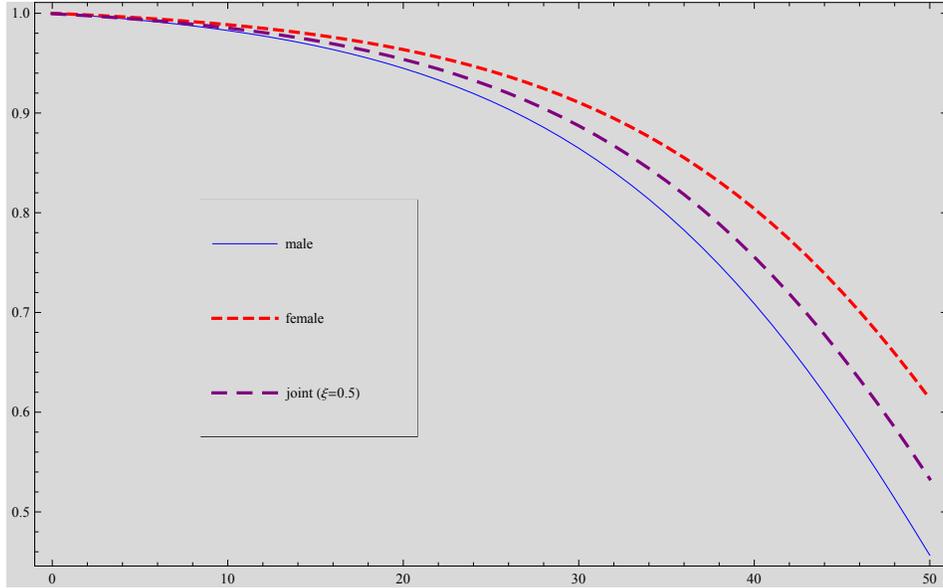


Figure 1: Survival probabilities relative to UK population, cohort 1950, initial age 35 (see values of Table 1) (source: HMD-2013).

## 5 Fair unisex mortality intensity

In this section, we assume that the mixed insurance portfolio is given a priori and we aim to determine the fair unisex mortality intensity that is consistent with the unisex fairness

principle (3) for that portfolio.

Regarding notation, we observe that the premium of a policy is just a function  $Prem(\cdot)$  of the underlying mortality intensity process. Therefore, assuming that the mortality intensities of male, females and unisex policyholders are given by  $\lambda^m$ ,  $\lambda^f$  and  $\lambda_\xi^u$  respectively, we shall naturally call

$$P_m := Prem(\lambda^m) \quad P_f := Prem(\lambda^f) \quad P_u := Prem(\lambda_\xi^u)$$

With this new notation, we can define what is the fair unisex mortality intensity that produces the unisex fair premium:

**Definition 5.1** (Fair unisex mortality intensity). *For a given portfolio of  $m$  male policyholders and  $n$  female policyholders, whose fair gender-based premiums are  $Prem(\lambda^m)$  and  $Prem(\lambda^f)$  respectively, we say that  $\lambda_{\xi^*}^u$  is a fair unisex mortality intensity if the corresponding unisex premium*

$$Prem(\lambda_{\xi^*}^u) = Prem(\xi^* \lambda^m + (1 - \xi^*) \lambda^f)$$

*is fair, i.e. it satisfies the unisex fairness principle (3):*

$$m \cdot Prem(\lambda^m) + n \cdot Prem(\lambda^f) = (m + n) \cdot Prem(\xi^* \lambda^m + (1 - \xi^*) \lambda^f). \quad (13)$$

*The weight  $\xi^*$  is said to be fair mortality mixing parameter.*

In other words, among the unisex mortality intensities of the family of Definition 4.1, the fair unisex mortality intensity computed with  $\xi^*$  is the only member that respects actuarial fairness.

Due to the closed-form expressions for the survival functions  $S^m(\cdot)$ ,  $S^f(\cdot)$  and  $S_\xi^u(\cdot)$ , the computation of the fair  $\xi^*$  is straightforward, once the mortality intensity processes  $\lambda^m$  and  $\lambda^f$  have been calibrated. As an illustrative example, in the following we will again consider the two most common life insurance products: pure endowment and term insurance. In the following we will assume that the discount factor is driven by the risk free interest rate  $r \geq 0$ :  $B(0, T) = e^{-rT}$ .

### Pure endowment

Consider the case in which  $m + n$  policyholders buy a pure endowment insurance contract which pays out a unitary payment if they survive the maturity date  $T$ . According to (13), the fair  $\xi^*$  results by setting:

$$m \cdot e^{-rT} \cdot S^m(0, T) + n \cdot e^{-rT} \cdot S^f(0, T) = (m + n) \cdot e^{-rT} \cdot S_{\xi^*}^u(0, T),$$

where the survival functions  $S^m(0, T)$  and  $S^f(0, T)$  are given by (8) (with  $i = m, f$  respectively), and the unisex survival function  $S_{\xi^*}^u(0, T)$  is given by (11). The above equation can be reduced to

$$m \cdot S^m(0, T) + n \cdot S^f(0, T) = (m + n) \cdot S_{\xi^*}^u(0, T) \quad (14)$$

from which  $\xi^*$  can be computed implicitly. Notice that in this case the fair  $\xi^*$  does not depend on the risk free rate  $r$ .

### Term insurance

Consider the case in which  $m + n$  policyholders buy a term insurance contract which pays out a unitary death benefit at the end of year of death if the contract holder does not survive the maturity date  $T$ . The fair  $\xi^*$  results by setting:

$$\begin{aligned} m \cdot \sum_{i=1}^N e^{-ri} (S^m(0, i-1) - S^m(0, i)) + n \cdot \sum_{i=1}^N e^{-ri} (S^f(0, i-1) - S^f(0, i)) \\ = (m + n) \cdot \sum_{i=1}^N e^{-ri} (S_{\xi^*}^u(0, i-1) - S_{\xi^*}^u(0, i)). \end{aligned} \quad (15)$$

Notice that in this case the fair  $\xi^*$  depends on the risk free rate  $r$ .

## 6 Numerical analysis

In this section, we analyze numerically how the fair  $\xi^*$  depends on the parameters of the model. In Section 6.1 we calibrate the parameters  $\mu_m, \mu_f, \sigma_m, \sigma_f$  for a number of cohorts

and ages. In Section 6.2, we first fix the value of  $\rho$  and determine, for all cohorts, the fair  $\xi^*$  for different insurance products and different durations; we analyze how  $\xi^*$  depends on the insurance product, on the duration and on the cohort; we then let  $\rho$  taking values in  $[-1, 1]$  and, for each cohort, we analyze how  $\xi^*$  depends on  $\rho$  for different insurance products. In Section 6.3, we find the sample correlation  $\rho$  between males' and females' mortalities, determine the fair  $\xi^*$  for different insurance products and compare across different cohorts.

## 6.1 Calibration procedure

We have taken cohort death rates from the Human Mortality Database (Last-modified: 06-May-2013) (HMD hereafter) for four different cohorts belonging to the UK population:

- cohort born in 1950 with initial age 35;
- cohort born in 1940 with initial age 45;
- cohort born in 1930 with initial age 55;
- cohort born in 1920 with initial age 65.

The calibration procedure is the following. For each cohort of initial age  $x$  and each gender we have extrapolated from the HMD twenty observed survival probabilities  ${}_t\hat{p}_x$ ,  $t = 1, \dots, 20$ ; then, we have calibrated the values of the parameters  $\mu_m, \mu_f, \sigma_m, \sigma_f$  that appear in the theoretical survival functions  $S_x^m(0, t)$  and  $S_x^f(0, t)$  given by (8) by minimizing the following mean square error

$$\frac{1}{20} \sum_{t=1}^{20} ({}_t\hat{p}_x - S_x^i(0, t))^2$$

for  $i = m, f$ . In all cases, the value of the initial observed intensity  $\lambda_x(0)$  is set equal to  $-\ln \hat{p}_x$ . This calibration procedure is standard and in the actuarial context it has been used to calibrate the OU intensity process in [7] and [9]. Table 1 reports the calibrated values of the parameters for the generations 1950 and 1940 for both genders, Table 2 reports those for cohorts 1930 and 1920. We see that in all cases  $\mu_m \neq \mu_f$ , therefore  $\lambda^u$  does not follow an OU process. In particular, except for cohort 1920,  $\mu_f < \mu_m$ .

	Cohort 1950, initial age $x = 35$		Cohort 1940, initial age $x = 45$	
	Female	Male	Female	Male
$\lambda_x(0)$	0.00075028	0.00112463	0.00207816	0.00329542
$\mu_x$	0.08001563	0.08171875	0.07561318	0.07731571
$\sigma_x$	0.00010305	0.00011789	0.00011809	0.00012212
Calibration Error	0.00000006	0.00000007	0.00000003	0.00000046

Table 1: Calibrated values and errors for males and females of cohort 1950 (initial age 35) and 1940 (initial age 45).

	Cohort 1930, initial age $x = 55$		Cohort 1920, initial age $x = 65$	
	Female	Male	Female	Male
$\lambda_x(0)$	0.00588629	0.00976351	0.016477	0.0313689
$\mu_x$	0.07452112	0.07609306	0.080625	0.06820313
$\sigma_x$	0.00011364	0.00012183	0.00009711	0.00009953
Calibration Error	0.00000085	0.00000111	0.00001412	0.00001689

Table 2: Calibrated values and errors for males and females of cohort 1930 (initial age 55) and 1920 (initial age 65).

## 6.2 The fair $\xi^*$

### 6.2.1 Fixed $\rho$

We have computed the fair  $\xi^*$  values for the pure endowment and the term insurance, for all cohorts and initial ages considered. For both products we have considered two different contract durations:  $T = 20$  and  $T = 30$  years. Common assumptions on the portfolios are  $\rho = 0.95$  and  $r = 0.03$ . We have considered different portfolio compositions by setting a fixed number of females  $n = 50$  and varying the number of males  $m$  in the range  $[10, 200]$ . Therefore, the proportion  $w = m/(m + n)$  of males in the portfolio varies from a minimum of 17% to a maximum of 80%. Tables 3, 4, 5 and 6 report the fair  $\xi^*$  for pure endowment (PE) and term insurance (TI) for  $T = 20, 30$  for cohorts 1950 (initial age 35), 1940 (initial age 45), 1930 (initial age 55), and 1920 (initial age 65), respectively.

$m$	$w = \frac{m}{m+n}$	PE, $T = 20$	PE, $T = 30$	TI, $T = 20$	TI, $T = 30$
10	0.167	0.1632	0.1626	0.1656	0.1640
20	0.286	0.2776	0.2792	0.2846	0.2826
50	0.500	0.4794	0.4900	0.5001	0.4992
100	0.667	0.6330	0.6550	0.6691	0.6707
150	0.750	0.7087	0.7377	0.7540	0.7575
200	0.800	0.7538	0.7874	0.8051	0.8099

Table 3: Fair  $\xi^*$  for pure endowment (PE) and term insurance (TI) with parameters:  $n = 50$ ,  $\rho = 0.95$ ,  $r = 0.03$ , generation born in 1950, initial age 35.

$m$	$w = \frac{m}{m+n}$	PE, $T = 20$	PE, $T = 30$	TI, $T = 20$	TI, $T = 30$
10	0.167	0.1614	0.1561	0.1631	0.1582
20	0.286	0.2764	0.2696	0.2808	0.2735
50	0.500	0.4828	0.4784	0.4951	0.4866
100	0.667	0.6429	0.6450	0.6643	0.6574
150	0.750	0.7227	0.7297	0.7497	0.7446
200	0.800	0.7706	0.7809	0.8012	0.7975

Table 4: Fair  $\xi^*$  for pure endowment (PE) and term insurance (TI) with parameters:  $n = 50$ ,  $\rho = 0.95$ ,  $r = 0.03$ , generation born in 1940, initial age 45.

$m$	$w = \frac{m}{m+n}$	PE, $T = 20$	PE, $T = 30$	TI, $T = 20$	TI, $T = 30$
10	0.167	0.1538	0.1372	0.1556	0.1429
20	0.286	0.2661	0.2409	0.2696	0.2501
50	0.500	0.4735	0.4410	0.4808	0.4555
100	0.667	0.6397	0.6107	0.6510	0.6277
150	0.750	0.7246	0.7008	0.7382	0.7185
200	0.800	0.7761	0.7567	0.7912	0.7746

Table 5: Fair  $\xi^*$  for pure endowment (PE) and term insurance (TI) with parameters:  $n = 50$ ,  $\rho = 0.95$ ,  $r = 0.03$ , generation born in 1930, initial age 55.

We observe the following:

- $\xi^*$ , that is the weight given to the males' mortality intensity in the unisex intensity, is never equal to the fraction of males in the portfolio  $w = m/(m+n)$ ; this means that

$m$	$w = \frac{m}{m+n}$	PE, $T = 20$	PE, $T = 30$	TI, $T = 20$	TI, $T = 30$
10	0.167	0.1346	0.1110	0.1394	0.1309
20	0.286	0.2367	0.1986	0.2447	0.2316
50	0.500	0.4352	0.3788	0.4479	0.4307
100	0.667	0.6048	0.5458	0.6197	0.6046
150	0.750	0.6955	0.6411	0.7109	0.6990
200	0.800	0.7521	0.7030	0.7675	0.7583

Table 6: Fair  $\xi^*$  for pure endowment (PE) and term insurance (TI) with parameters:  $n = 50$ ,  $\rho = 0.95$ ,  $r = 0.03$ , generation born in 1920, initial age 65.

the mix of the mortality intensities of males and females cannot be done using exactly the volume-related weights;

- nonetheless,  $\xi^*$  strongly depends on  $w$  and increases when  $w = m/(m+n)$  increases; this is obvious, because when there are more males in the portfolio, more weight must be given to the males mortality;
- $\xi^*$  is always smaller than  $w = m/(m+n)$  for the pure endowment; this is expected, because for the pure endowment the female is riskier than the male (given her higher survival probability for every duration), and therefore more weight should be given to the females' intensity; this is also consistent with the actuaries' current practice of giving more weight to the females' life table for the pure endowment;
- surprisingly,  $\xi^*$  is almost always smaller than  $w = m/(m+n)$  also for the term insurance; the only exceptions are for cohort 1950 and a sufficiently high number of males in the portfolio (greater than 50% of the portfolio for  $T = 20$ , greater than 2/3 of the portfolio for  $T = 30$ ); this is not expected, because for the term insurance the male is riskier than the female (given his higher death probability for every duration); therefore, this is not consistent with the actuaries' current practice; this result seems to suggest that, if the number of females is sufficiently high, the insurer must give more weight than the volume-related proportion to the females mortality, no matter what the insurance product is;

- for each cohort, portfolio size and duration,  $\xi^*$  is greater for the term insurance than for the pure endowment; this is expected, because with the term insurance more weight must be given to the males mortality than with the pure endowment; this is clearly consistent with the current practice of giving more weight to the males' life table for the term insurance and less weight to the males' life table for the pure endowment;
- with some exceptions (especially cohort 1950 and pure endowment),  $\xi^*$  is generally higher for shorter duration ( $T = 20$ ) and smaller for longer duration ( $T = 30$ ); this result is meaningful and seems to indicate that when the duration is high the importance given to females should increase; this is maybe due to the fact that females live longer, so the share of males  $m/(m+n)$  decreases in the long run (as more males die);
- the difference between  $\xi^*$  and the volume-related proportion  $w = m/(m+n)$  is higher for older cohorts and ages, and smaller for younger cohorts and ages: the average relative deviation is about 10-20% for cohort 1920-age 65, about 5-10% for cohort 1930-age 55, about 3-4% for cohort 1940-age 45 and is about 1-2% for cohort 1950-age 35,  $\xi^*$  being (almost) always smaller than  $w$ ; this result indicates that the older the cohort and the age, the higher the weight that should be allocated to females; the explanation is similar to that given in the previous point: females live longer, so the share of males  $m/(m+n)$  decreases in the long run (as more males die), and this happens with higher severity with older cohorts than with younger ones, also because the older cohorts have a higher initial age than younger cohorts; furthermore, this also reflects the higher improvements in the males' mortality from older to younger cohorts.

### 6.2.2 Varying $\rho$

In Tables 3, 4, 5, 6, the correlation coefficient  $\rho$  was set to 0.95. However,  $\xi^*$  in principle may depend also on  $\rho$  and, if this is the case, the insurer should take into account this dependence in pricing unisex tariffs. Thus, we have investigated the dependence of  $\xi^*$  on  $\rho$ . Figures 2 – 5 show, for the four cohorts considered, the effect of  $\rho$  on  $\xi^*$  for the pure endowment and the term insurance. Common assumptions on the portfolios are  $m = n = 50$ , that implies  $w = m/(m+n) = 0.5$ ,  $T = 20$  and  $r = 0.03$ . For the duration  $T = 30$  we would get similar figures.

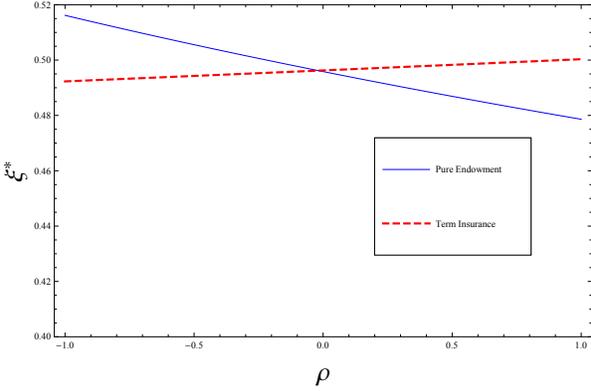


Figure 2.  $\xi^*$  versus  $\rho$ , cohort 1950.

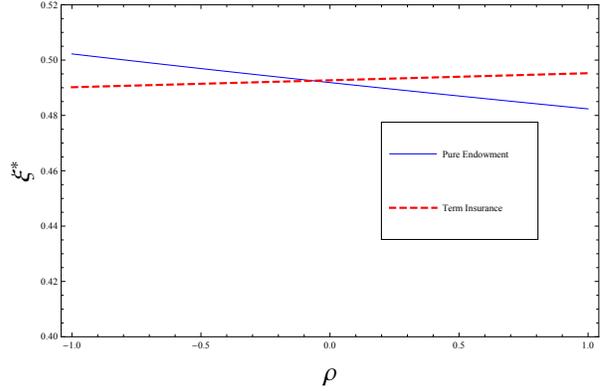


Figure 3.  $\xi^*$  versus  $\rho$ , cohort 1940.

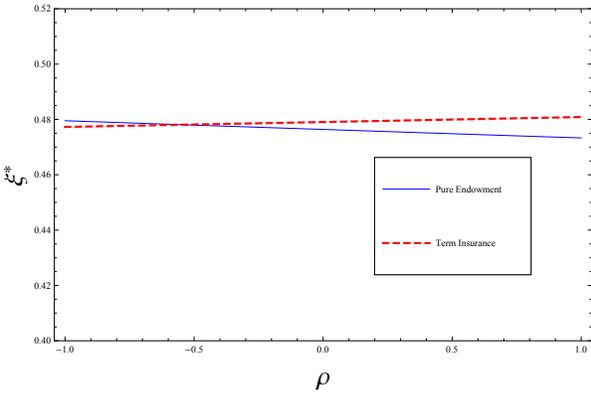


Figure 4.  $\xi^*$  versus  $\rho$ , cohort 1930.

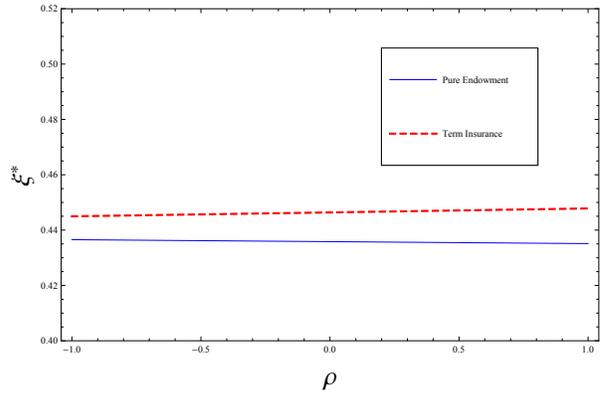


Figure 5.  $\xi^*$  versus  $\rho$ , cohort 1920.

We observe the following:

- The main result is that  $\xi^*$  can depend on  $\rho$ ; this implies that the knowledge of  $\rho$  should not be ignored by insurers in charging unisex tariffs, at least for some cohorts.
- For cohorts 1950 and 1940,  $\xi^*$  is slightly decreasing with  $\rho$  for the pure endowment, while it is slightly increasing for the term insurance. For cohorts 1930 and 1920,  $\xi^*$  seems almost unaffected by the varying  $\rho$ .
- A possible explanation for the decreasing trend for the pure endowment can be the following: the maximum weight put in the females for the pure endowment is with  $\rho = 1$ ; when  $\rho$  decreases the behaviour of  $\lambda^m$  is less correlated to  $\lambda^f$ , it cannot be

explained with  $\lambda^f$ , so it is necessary to give more weight to the component  $\lambda^m$  and  $\xi^*$  increases;

- A similar explanation holds for the increasing trend for the term insurance: the maximum weight put in the males for the term insurance is with  $\rho = 1$ ; when  $\rho$  decreases the behaviour of  $\lambda^f$  is less correlated to  $\lambda^m$ , it cannot be explained with  $\lambda^m$ , so it is necessary to give more weight to the component  $\lambda^f$  and  $\xi^*$  decreases;
- For cohort 1920,  $\xi^*$  is less sensible to  $\rho$  than for other cohorts; in addition,  $\xi^*$  is always higher for the term insurance than for the pure endowment; this last result is expected, because (as mentioned earlier in this section) with the term insurance more weight must be given to the males mortality than with the pure endowment;
- For the other cohorts,  $\xi^*$  is higher for the term insurance than for the pure endowment only for  $\rho > \hat{\rho}$ , where  $\hat{\rho}$  is about 0 for 1940 and 1950 and is about -0.5 for 1930; there is no plausible explanation for  $\xi^*$  being lower for term insurance than for pure endowment, and we deduce that this weird inequality happens only in the presence of unrealistic values of  $\rho$ .

### 6.3 The fair $\xi^*$ with the sample correlation $\rho$

The results of Section 6.2.2 indicate the relevance of  $\rho$  on the fair unisex mortality intensity. It is then important to calculate the value of  $\rho$ , which is in general not necessary for the knowledge of the marginal mortality intensities of males and females. In this section, we estimate the sample correlation  $\rho$  and present the corresponding values of the fair  $\xi^*$ . The estimation procedure is presented in the Appendix.

Table 7 reports the estimated correlation coefficient  $\rho$  for the four cohorts considered. The

Cohort	1950	1940	1930	1920
$\rho$	0.855	0.832	0.876	0.948

Table 7: Estimated  $\rho$  for the four cohorts considered.

estimated  $\rho$  is smaller than the fixed  $\rho = 0.95$  selected in Section 6.2. Tables 8, 9, 10 and

11 report the fair  $\xi^*$  for the four insurance products considered for cohorts 1950, 1940, 1930 and 1920, respectively.

$m$	$w = \frac{m}{m+n}$	PE, $T = 20$	PE, $T = 30$	TI, $T = 20$	TI, $T = 30$
10	0.167	0.1633	0.1624	0.1655	0.1637
20	0.286	0.2780	0.2789	0.2844	0.2821
50	0.500	0.4810	0.4899	0.4997	0.4983
100	0.667	0.6357	0.6551	0.6685	0.6694
150	0.750	0.7121	0.7380	0.7533	0.7560
200	0.800	0.7577	0.7879	0.8044	0.8083

Table 8: Fair  $\xi^*$  for pure endowment (PE) and term insurance (TI) with parameters:  $n = 50$ ,  $\rho = 0.855$ ,  $r = 0.03$ , generation born in 1950, initial age 35.

$m$	$w = \frac{m}{m+n}$	PE, $T = 20$	PE, $T = 30$	TI, $T = 20$	TI, $T = 30$
10	0.167	0.1615	0.1560	0.1631	0.1581
20	0.286	0.2768	0.2696	0.2807	0.2733
50	0.500	0.4839	0.4785	0.4948	0.4861
100	0.667	0.6448	0.6453	0.6638	0.6566
150	0.750	0.7252	0.7302	0.7490	0.7436
200	0.800	0.7734	0.7816	0.8005	0.7964

Table 9: Fair  $\xi^*$  for pure endowment (PE) and term insurance (TI) with parameters:  $n = 50$ ,  $\rho = 0.832$ ,  $r = 0.03$ , generation born in 1940, initial age 45.

Comparing Tables 8, 9, 10 and 11 with Tables 3, 4, 5 and 6, we see that for cohort 1920 the values of  $\xi^*$  are almost the same, that is expected given that the estimated  $\rho$  is almost 0.95 and that for that cohort  $\xi^*$  is almost constant in  $\rho$ . For the other cohorts, the value of  $\xi^*$  with the estimated  $\rho$  is slightly higher than with  $\rho = 0.95$  for the PE, and is slightly lower than with  $\rho = 0.95$  for the TI. Although the sign in the difference is expected, we also notice that the difference is negligible. This indicates that the error made by a misspecification of  $\rho$  should not be relevant.

$m$	$w = \frac{m}{m+n}$	PE, $T = 20$	PE, $T = 30$	TI, $T = 20$	TI, $T = 30$
10	0.167	0.1539	0.1372	0.1556	0.1428
20	0.286	0.2662	0.2409	0.2695	0.2501
50	0.500	0.4737	0.4411	0.4806	0.4553
100	0.667	0.6402	0.6107	0.6507	0.6274
150	0.750	0.7252	0.7009	0.7504	0.7181
200	0.800	0.7767	0.7568	0.7908	0.7741

Table 10: Fair  $\xi^*$  for pure endowment (PE) and term insurance (TI) with parameters:  $n = 50$ ,  $\rho = 0.876$ ,  $r = 0.03$ , generation born in 1930, initial age 55.

$m$	$w = \frac{m}{m+n}$	PE, $T = 20$	PE, $T = 30$	TI, $T = 20$	TI, $T = 30$
10	0.167	0.1346	0.1110	0.1394	0.1309
20	0.286	0.2367	0.1986	0.2447	0.2316
50	0.500	0.4352	0.3788	0.4477	0.4307
100	0.667	0.6048	0.5458	0.6197	0.6046
150	0.750	0.6956	0.6411	0.7109	0.6990
200	0.800	0.7521	0.7030	0.7675	0.7583

Table 11: Fair  $\xi^*$  for pure endowment (PE) and term insurance (TI) with parameters:  $n = 50$ ,  $\rho = 0.948$ ,  $r = 0.03$ , generation born in 1920, initial age 65.

## 7 Concluding remarks

The actuarial implications of the EU Gender Directive (European Union Directive 2004/113/EC) for life insurance products are enormous. In the lack of theoretical models, the current practice adopted by actuaries for the calculation of the unisex premium seems to be a weighted average of the gender-specific life tables, with proper corrections linked to the type of insurance product. This paper proposes a theoretical model for the pricing of the unisex premium of a life insurance product. The main contributions are twofold. First, we formulate the unisex actuarial fairness principle and define the corresponding unisex actuarially fair premium. Second, we introduce the fair unisex mortality intensity, that is the unisex intensity underlying the unisex fair premium.

The numerical application is rich of interesting results. We find that, in accordance with

the current actuarial practice, in the fair unisex mortality intensity more weight must be given to the males' intensity for the term insurance and, opposite, more weight must be given to the females' intensity for the pure endowment. A surprising result is that if the number of females in the portfolio is sufficiently high, then the weight to be given to the females' intensity is higher than the volume-related weight not only for the pure endowment, but also for the term insurance. A comparison across different cohorts and ages shows that the older the cohort and the age, the higher the weight to allocate to the females' intensity.

Up to our knowledge, the unisex fairness principle, the unisex fair premium as well as the fair unisex mortality intensity are novel in the actuarial literature and could be useful tools for the life insurance company. The knowledge of the fair unisex stochastic mortality model should facilitate several tasks of the life insurance company including pricing, reserving, profit testing and computation of solvency capital requirements (SCR) for unisex portfolios. The use of the present model for the calculation of Solvency II SCR for mixed life insurance portfolios is in the agenda for future research.

Finally, this paper leaves ample space for further research. Possible extensions are: the introduction of a stochastic discount factor; the introduction of safety loading and commissions; the derivation of  $\xi^*$  for other insurance products; a stochastic proportion of males in the portfolio.

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## Appendix

In order to calibrate the correlation coefficient  $\rho$ , we use the fact that

$$dW^m(t) dW^f(t) = \rho dt,$$

and we express  $d\lambda^f(t)$  with two orthogonal Brownian motions  $dW^m(t)$  and  $dW^{\perp m}(t)$ :

$$\begin{aligned} d\lambda^m(t) &= \mu_m \lambda^m(t) dt + \sigma_m dW^m(t), \\ d\lambda^f(t) &= \mu_f \lambda^f(t) dt + \sigma_f \left( \rho dW^m(t) + \sqrt{1 - \rho^2} dW^{\perp m}(t) \right). \end{aligned} \quad (16)$$

From (16) it is easy to see that

$$d\lambda^m(t) \cdot d\lambda^f(t) = \rho \sigma_m \sigma_f dt = E(d\lambda^m(t) \cdot d\lambda^f(t)),$$

and

$$E(d\lambda^m(t)) \cdot E(d\lambda^f(t)) = 0, \quad Var(d\lambda^m(t)) = \sigma_m^2 dt, \quad Var(d\lambda^f(t)) = \sigma_f^2 dt. \quad (17)$$

Therefore,

$$Cov(d\lambda^m(t), d\lambda^f(t)) = E(d\lambda^m(t) \cdot d\lambda^f(t)) - E(d\lambda^m(t)) \cdot E(d\lambda^f(t)) = \rho \sigma_m \sigma_f dt, \quad (18)$$

and, using (17) and (18), the Pearson coefficient of  $d\lambda^m(t)$  and  $d\lambda^f(t)$  is:

$$\frac{Cov(d\lambda^m(t), d\lambda^f(t))}{\sqrt{Var(d\lambda^m(t))} \sqrt{Var(d\lambda^f(t))}} = \rho.$$

Thus,  $\rho$  coincides with the Pearson coefficient of  $d\lambda^m(t)$  and  $d\lambda^f(t)$ , and this result allows its empirical computation from the data.