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Theodoros M. Diasakos

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A Simple Characterization of Dynamic Completeness in Continuous Time

Theodoros M. Diasakos*

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Abstract

I establish a necessary and sufficient condition for the securities' market to be dynamically-complete in a single-commodity, pure-exchange economy with many Lucas' trees whose dividends are geometric Brownian motions. Even though my analysis is based upon the representative-agent version of this economy, the condition depends neither on the utility function of the representative agent, nor on the functional form of her endowment. As a consequence, it characterizes dynamic completeness in this economy even in the presence of many heterogenous agents.

Keywords: Dynamically-Complete Markets, Continuous Time, General Equilibrium.
JEL Classification Numbers: G10, G12.

*Collegio Carlo Alberto, Moncalieri (TO), Italy. *theodoros.diasakos@carloalberto.org*

1 Introduction

For the most part of the theoretical continuous-time financial economics literature, the workhorse has been some analogue of the model in Lucas [18]. In its purest form, it depicts a one-commodity, pure-exchange economy with identical price-taking consumers, in which economic activity occurs over a time-interval. The good is produced by distinct units whose productivity fluctuates stochastically, their usual interpretation being that of Lucas trees. Namely, a crop is growing stochastically on different trees via a production process that is entirely exogenous: no resources are utilized and there is no possibility of affecting the output of any tree at any time.

Even though commonly endowed with the generated filtration, the individuals cannot observe the actual productivity shocks. Instead, they monitor the crop on the trees, whose magnitude plays the role of an information process. In the basic model, the agents use this information to trade continuously and frictionlessly a given set of perfectly divisible securities. This consists of one security, in positive net supply, for each productive unit representing one equity share (termed "stock") in that tree. There is also a promissory note (termed "bond"), in zero net supply, paying one unit of the good with certainty.

In the equilibrium of this economy, the price of the typical security is the current expectation of its future dividends valued at the representative agent's marginal rate of substitution between consumption at the dividend-collection date and the present. Derivations have been provided by a number of seminal papers and for different versions of the model (see the next section for more details), which have been then used extensively in the literature to price other financial assets, such as derivative securities, and to identify the optimal consumption and portfolio policies. Yet, what has not been thus far determined (at least not to a satisfactory degree of generality with respect to the economic primitives) is under what conditions the dynamics of the equilibrium price processes with respect to the underlying stochastic process allow the securities' market to be dynamically complete.

As a property of the underlying securities' market, dynamic completeness

has had a crucial role in the continuous-time finance literature. It features, within the vast majority of the pertinent papers, as a central assumption for proving the existence of the equilibrium pricing process itself. Building upon seminal works (see Harrison and Kreps [14], Duffie and Huang [10], Duffie [9], Mas Collel [19] or Mas Collel and Richard [20] among others), the usual way of obtaining financial equilibria in continuous-time economies has been to compute an Arrow-Debreu equilibrium and use the associated consumption process as a pricing kernel in order to construct candidate equilibrium pricing processes for the traded securities. Yet, to ensure that the Arrow-Debreu equilibrium allocation is indeed implementable by dynamically (and sufficiently rapidly) trading the given set of securities, their market needs to be dynamically complete. And this is nearly always impossible to verify since the candidate pricing processes are either not given in closed form or presented as conditional expectations.

Of course, this role of dynamic completeness is a matter of choice regarding the strategy of proof, and one that is not binding when it comes to establishing existence of equilibrium in representative agent economies.[1] Yet, as a matter of economics, dynamic completeness is fundamental for the justification of the representative agent herself and for using her equilibrium pricing kernel to price financial assets that are not amongst the primitives of the model. If the pricing process of the underlying securities is dynamically complete, then options and other derivative securities can be uniquely priced by arbitrage arguments and replicated by trading the underlying securities. In the absence of dynamic completeness, however, this is no longer the case; arbitrage considerations do not suffice to determine unique option prices and replication is not possible.

Given a financial environment, therefore, it is important to be able to associate dynamic completeness with at least some of the economic primitives in a manner that remains unambiguously verifiable and holds generically across the space of these primitives. Restricting attention to the case in

[1] In fact, Raimondo [25] as well as Anderson and Raimondo [2] establish existence of equilibrium in a representative agent economy without having to assume even potential dynamic completeness.

which the typical component of the production process follows a geometric Brownian motion, this is precisely the contribution of the present paper with respect to the securities' market and the economy described above.

In this economy, when the securities' market is potentially dynamically complete, it is in fact dynamically complete if and only if the dispersion matrix (the matrix of factor loadings) of the Brownian process is nonsingular (Theorem 1). Even though rather widely asserted in the relevant literature, this claim has not been shown explicitly before. More importantly perhaps, it is universal with respect to the specification of the representative agent's utility function and her endowment. It applies, therefore, to the given economy even when there are many individuals with heterogenous preferences.

The remainder of the paper is organized as follows. In the next section, the model I study is placed in the context of the pertinent literature. Section 3 presents and analyzes the main result while Section 4 relates it to similar conditions that have been postulated elsewhere. The proof of the theorem, as well as the material in support of other claims I make throughout the paper, are in Section 5.

2 Theoretical Foundations

Consider a one-commodity, pure-exchange economy with identical price-taking consumers, in which economic activity occurs over a time-interval $[0,T] \subseteq \mathbb{R}_+$.[2] The consumption good is produced by $N \in \mathbb{N}^*$ Lucas trees whose productivity fluctuates stochastically according to a K-dimensional ($K \geq N$) standard Brownian motion $\beta = \{\beta(\omega,t) : t \in [0,T]\}_{\omega\in\Omega}$ defined on a complete probability space $(\Omega, \mathcal{F}, \pi)$. This is meant to describe the exogenous uncertainty about productivity in the sense that the sample paths in the collection $\{\beta(\omega,[0,T])\}_{\omega\in\Omega}$ completely specify all the distinguishable events.

Even though endowed with the generated filtration $\{\mathcal{F}_t : t \in [0,T]\}$, the

[2]The theoretical backdrop of the financial environment for which the present result applies is the same as that in Diasakos [8]. This section gives an outline of the more detailed description provided there.

agents cannot observe β directly. Instead of the actual productivity shocks, they monitor the crop on the trees, depicted by the N-dimensional process Y, which is a function of the process $\mathcal{I} = \{\beta(\omega, t), t\}_{(\omega,t)\in\Omega\times[0,T]}$ and whose component processes $Y_1, \ldots, Y_N$ represent the current amount of the consumption good on the respective tree. Of course, the evolution of Y over time depends upon β in a nonpredictable fashion, being adapted to the given filtration.

The trading structure consists of $N+1$ securities: a zero-coupon bond, depicted as the zeroth security, and stocks indexed by $n \in \{1, \ldots, N\}$. Finally, individual preferences are such that the representative agent has some von-Neumann Morgenstern utility function over instantaneous consumption, $u : \mathbb{R}_{++} \mapsto \mathbb{R}$, which is twice continuously-differentiable, strictly increasing, and concave everywhere in its domain.

The underlying informational structure being a filtration, the choice of numeraire here is essentially arbitrary because the equilibrium market-clearing condition will depend only on the prices of the securities relative to the price of consumption, and will do so node (ω, t) by node (ω, s), for $s \neq t$.[3] We may choose, therefore, consumption as the numeraire and set its price at $P_c(\omega, t) = 1 \ \forall (\omega, t) \in \Omega \times [0, T]$. What matters then is the equilibrium price process of the typical stock relative to that of the bond: $p_n(\omega, t) = \frac{P_n(\omega,t)}{P_0(\omega,t)}$.[4]

[3]Recall that each $\omega \in \Omega$ is a complete description of the uncertain environment. As such, it gets predetermined exogenously and remains fixed throughout time. What changes with time is the path of realizations for the underlying stochastic process that generates the filtration $\{\mathcal{F}_t : t \in [0, T]\}$. Being a K-dimensional standard Brownian motion, its component processes $\beta_1, \ldots, \beta_K$ are independent, one-dimensional Brownian motions with zero drift and unit variance so that the process changes here in increments such that, for all $0 \leq s < t \leq T$, $\beta(\omega, t) - \beta(\omega, s)$ is independent of $\mathcal{F}_s(\omega)$ and distributed $\mathcal{N}(\mathbf{0}, (t-s)\,\mathrm{I}_K)$. A given ω determines, therefore, the Brownian path $\beta(\omega, [0, T])$. And since this path has been drawn by nature before the economic activity even starts, the equilibrium market-clearing conditions need to apply only along the path; along every possible path, of course, but not across paths. As a consequence, and given that only relative prices matter in equilibrium, it is without loss of generality to normalize such that the price of one of the traded entities is 1 throughout every path.

[4]This is the price of the typical stock measured in bond units. The absolute price of the bond $P_0(\omega, t)$ being strictly positive everywhere on $\Omega \times [0, T]$, this normalization amounts to measuring everything in bond units so that the bond itself is turned into a numeraire money market account. The self-financing strategy is to buy-and-hold one unit

In what follows, I will use the closed form solution for $p_n(\omega, t)$ as this has been provided by two related strands of the literature. The first assumes that the crop on the trees is ripe for consumption only at a finite terminal date T. At any intermediate time $t \in [0, T)$, the agent consumes some exogenously-given deterministic endowment flow (see, for example, Raimondo [25] as well as Anderson and Raimondo [2]) or nothing at all (as in Bick [4]-[5] but also He and Leland [15]). Letting W denote the representative agent's wealth process (in units of consumption), we have then

$$p_n(\omega, t) = \frac{P_n(\omega, t)}{P_0(\omega, t)} = \frac{\mathbb{E}_\pi\left[u'\left(W\left(\mathcal{I}(\omega, T)\right)\right) D_n\left(\mathcal{I}(\omega, T)\right) | \mathcal{F}_t\right]}{\mathbb{E}_\pi\left[u'\left(W\left(\mathcal{I}(\omega, T)\right)\right) | \mathcal{F}_t\right]} \tag{1}$$

In Bick [5], Raimondo [25], as well as Anderson and Raimondo [2], the production, consumption, information, trading, and preferences structures but also the dividends' specification are exactly as in the present analysis. The same is true, apart for a much more general dividend specification, regarding Bick [4] as well as He and Leland [15], two models with no real differences between them. Either assumes $N = K = 1$ and that the representative agent has no endowment - other than the net supply of the stock (which can be viewed as the market portfolio), - two restrictions present also present in Bick [5]. As a consequence, in all three papers consumption takes place only at the final date. By contrast, Raimondo [25] as well as Anderson and Raimondo [2] assume that the agent is endowed with a deterministic flow rate of consumption during the interval $[0, T)$ and with a lump sum at T, which may be stochastic (a continuous function of the terminal-date realization of the underlying Brownian process). Nevertheless, in all five papers, the equilibrium relative price of the typical stock is given by the fundemental equation (1).

The second approach in the literature has been to consider the actual continuous-time extension of the setting in Lucas [18], granting the agent continuous access to the crop so that her consumption can be financed by the trees' payoffs at all times while T may be infinite. The equilibrium relative price of the nth risky security is then essentially the flow-analogue

of the bond. The account pays no interest since its value is constant at one unit.

of that in (1):

$$p_n(\omega, t) = \frac{\mathbb{E}_\pi \left[\int_t^T u'(W(\mathcal{I}(\omega, s)), s) D_n(\mathcal{I}(\omega, s)) \, \mathrm{d}s | \mathcal{F}_t \right]}{\mathbb{E}_\pi \left[\int_t^T u'(W(\mathcal{I}(\omega, s)), s) \, \mathrm{d}s | \mathcal{F}_t \right]} \tag{2}$$

From all of the papers in this strand, the most well-known is Cox et al. [7], probably the most seminal study of the continuous-time, single-good economy with identical agents and Lucas trees. Lucas [18] considered an infinite-horizon, discrete-time, single- and perishable-good, pure-exchange economy with several trees in which a representative agent with state- and time-independent utility for instantaneous consumption and no endowment (other than the trees) has continuous access to the trees' output, so that intermediate consumption is financed by the trees' dividends.

Cox et al. [7] present the continuous-time analogue of this model, enhancing it to include production. As before, an underlying stochastic process generates shocks to the productivity of the trees. Yet, the trees' productivity is now influenced also by the representative agent who has continuous access to the trees' output, consuming some and reinvesting the rest in the production process. The authors consider in addition a more general preference structure along, however, with a more restricted trading one. The agent may have now state- and time-dependent preferences for instantaneous consumption while there is a dynamically-complete securities market in which a full set of Arrow-Debreu contingent claims are traded (each available in zero net supply).

Allowing for time- but not state-dependence, the representative agent of Cox et al. [7] seeks to maximize the current expectation of the entire future utility flow, $\mathbb{E}_\pi \left[\int_t^T u(W(\mathcal{I}(\omega, s)), s) \, \mathrm{d}s | \mathcal{F}_t \right]$. In this case, the equilibrium price of any real asset relative to that of the zero-coupon bond is given by (2). The same pricing formula can be found also in Merton [22]-[23], Cochrane et al. [6], Martin [21], Duffie and Zame [13] (see Theorem 1 and the subsequent discussion in Section 5), Karatzas et al. [17] (Corollary 10.4), Riedel [26] (Theorem 2.1), and Wang [27] (Equation 2.4).

It should be pointed out also that, even when the individuals in the

economy have non-identical preferences for consumption, the pricing formula takes still the same basic form as in (1)-(2). The only difference is that the individual marginal utilities are now taken at the equilibrium consumptions of the agents, which are determined endogenously as part of the equilibrium (see, for instance, Duffie and Zame [13] or Anderson and Raimondo [1]).

To enable the analytical manipulation of the fundamental pricing equations in (1)-(2), I will restrict attention to the case in which the typical component of the production process follows a geometric Brownian motion: $Y_n(\mathcal{I}(\omega,t)) = e^{\mu_n t + \sigma_n^\intercal \beta(\omega,t)}$, both the drift $\mu_n \in \mathbb{R}$ and the instantaneous covariance matrix $\sigma_n \sigma_n^\intercal \in \mathbb{R}^{K\times K}$ being constants. This is a widely-used specification, both in the theoretical as well as empirical literature, which allows the derivative $\frac{\partial p_n(\omega,t)}{\partial \beta_k(\omega,t)}$ to be recovered from the current information on future dividends in a very straightforward way. More importantly perhaps for the purposes of the current study, it greatly facilitates exposition as it allows us to obtain a characterization of dynamic completeness with respect to the pricing process in (1) which is also valid for that in (2). Indeed, in what follows and for reasons of expositional clarity, my analysis refers to the fundamental pricing equation (1) even though, as I show in Section 5, it readily carries through also to the pricing equation (2).

For any $\omega \in \Omega$, therefore, at all intermediate dates $t \in [0,T)$, the dividends of the $N+1$ securities are zero while the representative agent's endowment process is deterministic. At the terminal date, however, the dividends will be given by $D_0(\mathcal{I}(\omega,T)) = 1$ and $D_n(\mathcal{I}(\omega,T)) = e^{\mu_n T + \sigma_n^\intercal \beta(\omega,T)}$ for $n = 1,\ldots,N$. The representative agent's endowment process, moreover, will be given by some deterministic function $e : [0,T) \mapsto \mathbb{R}_+$ at all times prior to the terminal date and by $\rho(\mathcal{I}(\omega,T))$, for some continuous function $\rho : \mathbb{R}^K \times \{T\} \mapsto \mathbb{R}_+$, at the end. The agent's wealth (equivalently, her equilibrium consumption) equals, therefore, her deterministic endowment during the intermediate period and

$$W(\omega,T) = \rho(\mathcal{I}(\omega,T)) + \sum_{n=1}^{N} D_n(\mathcal{I}(\omega,T))$$

at the end. She also has an additively-separable, time-independent utility function which, for a measurable with respect to the Brownian filtration consumption function $c : \Omega \times [0,T] \to \mathbb{R}_{++}$, is given by

$$U\left(c\left(\mathcal{I}\left(\omega,t\right)\right)\right) = \mathbb{E}_{\pi}\left[\int_{t}^{T} v\left(c\left(\mathcal{I}\left(\omega,s\right)\right)\right)\mathrm{d}s + u\left(c\left(\mathcal{I}\left(\omega,T\right)\right)\right)|\mathcal{F}_{t}\right] \tag{3}$$

for some instantaneous utility functions $v, u : \mathbb{R}_{++} \mapsto \mathbb{R}$ that are everywhere twice continuously-differentiable, strictly increasing, and strictly concave.

The corresponding equilibrium pricing process has been derived explicitly by Raimondo [25], in terms of the agent's utility function, her terminal-period endowment, and the current realization $\beta\left(\omega,t\right)$ of the Brownian vector:[5]

$$\begin{aligned}
P_{n}\left(\omega,t\right) &= \mathbb{E}_{\pi}\left[u'\left(W\left(\mathcal{I}\left(\omega,T\right)\right)\right)D_{n}\left(\mathcal{I}\left(\omega,T\right)\right)|\mathcal{F}_{t}\right] \\
&= \int_{\mathbb{R}^{K}} u'\left(W\left(\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)\right)e^{\mu_{n}T+\sigma_{n}^{\intercal}\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}\mathrm{d}\Phi\left(\mathbf{x}\right) \\
P_{0}\left(\omega,t\right) &= \mathbb{E}_{\pi}\left[u'\left(W\left(\mathcal{I}\left(\omega,T\right)\right)\right)|\mathcal{F}_{t}\right] = \int_{\mathbb{R}^{K}} u'\left(W\left(\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)\right)\mathrm{d}\Phi\left(\mathbf{x}\right)
\end{aligned}$$

Here, the quantities

$$\begin{aligned}
W\left(\mathcal{I}\left(\omega,t\right),\mathbf{x}\right) &= \rho\left(\beta\left(\omega,t\right)+\sqrt{T-t}\mathbf{x}\right)+\sum_{n=1}^{N} D_{n}\left(\mathcal{I}\left(\omega,t\right),\mathbf{x}\right) \\
D_{n}\left(\mathcal{I}\left(\omega,t\right),\mathbf{x}\right) &= e^{\mu_{n}T+\sigma_{n}^{\intercal}\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}
\end{aligned} \tag{4}$$

depict, respectively, the terminal realizations of the agent's wealth and of the nth dividend, conditional on the current Brownian realization and on its future increment $\beta\left(\omega,T\right)-\beta\left(\omega,t\right)=\sqrt{T-t}\mathbf{x}$, with $\mathbf{x}\sim\mathcal{N}\left(\mathbf{0},\mathrm{I}_{K}\right)$ and $\Phi\left(\cdot\right)$ being the standard K-dimensional Normal cumulative distribution function.

Notice that both of the last two quantities above as well as all expectations henceforth are $\mathcal{F}_{t}$-conditional. My focus will be on the derivative of

[5] See Theorem 1 in Raimondo [25] but also Theorem 2.1 in Anderson and Raimondo [2]. The price of consumption is given by $P_{C}\left(\omega,t\right)=v'\left(e\left(t\right)\right)$ for $t\in[0,T)$ and $P_{C}\left(\omega,T\right)=u'\left(W\left(\mathcal{I}\left(\omega,T\right)\right)\right)$. Recall, however, that we have normalized setting $P_{C}\left(\omega,t\right)=1\ \forall\left(\omega,t\right)\in\Omega\times[0,T]$; hence, all other prices are measured in units of marginal utility (utils per unit of consumption).

the typical relative price

$$p_n(\omega,t) = \frac{P_n(\omega,t)}{P_0(\omega,t)} = \frac{\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathcal{I}(\omega,t),\mathbf{x}\right)\right)D_n\left(\mathcal{I}(\omega,t),\mathbf{x}\right)\right]}{\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(\mathcal{I}(\omega,t),\mathbf{x}\right)\right)\right]}$$

with respect to changes in $\beta_k(\omega,t)$, the current realization of the typical Brownian motion. This is given by

$$\frac{\partial p_n(\omega,t)}{\partial \beta_k(\omega,t)} = \frac{1}{P_0(\omega,t)}\left[\frac{\partial P_n(\omega,t)}{\partial \beta_k(\omega,t)} - p_n(\omega,t)\frac{\partial P_0(\omega,t)}{\partial \beta_k(\omega,t)}\right] \tag{5}$$

3 Dynamic Completeness

In an Arrow-Debreu economy, the agents may shift consumption across states and time by trading a complete set of contingent claims. When they are constrained to trade a given set of securities, however, the market is said to be dynamically complete if the agents can still achieve any consumption allocation that would be feasible if there were instead a complete set of Arrow-Debreu contingent claims. Under continuous-time trading, when the information about the state of the world is revealed through a stochastic process, this may be possible by trading a given finite set of securities rapidly enough. In particular, when the underlying uncertainty is driven by Brownian motions, a necessary (yet by no means sufficient) condition for this to happen is that the securities market is potentially dynamically-complete: i.e., the number of securities exceeds that of independent Brownian motions by at least one.[6]

Hence, in the economy described previously, let $N = K$. It is well-known that, in the presence of a money market account, dynamic completeness is equivalent to the dispersion matrix of absolute prices $\left[\frac{\partial P_n(\omega,t)}{\partial \beta_k(\omega,t)}\right]_{(n,k)\in\{1,\ldots,K\}^2}$ having almost everywhere rank equal to K, the number of the sources of risk (see, for example, Sections 4.1-4.4 and Theorem 5.6 in Nielsen [24] or Chapter 6 in Duffie [9]). Here, given our normalization, the prices of the risky-securities are already expressed in bond units so that the zero-

[6] When the underlying information process is not Brownian, the required number of securities may be larger.

coupon bond is itself a numeraire money market account.[7] Hence, dynamic completeness is equivalent to the $K \times K$ dispersion matrix of relative prices

$$J_p(\omega, t) = \left[\frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)}\right]_{(n,k)\in\{1,\ldots,K\}^2}$$

having almost everywhere on $\Omega \times [0, T)$ rank equal to K, the dimension of the Brownian process. As it turns out, this is in turn equivalent to the matrix of factor loadings being nondegenerate.[8]

Theorem 1 *Let the securities market be potentially dynamically complete ($N = K$). The following are equivalent.*

(i) The market is in fact dynamically complete.

(ii) Σ is nonsingular.

As a desirable feature of the economic environment under study, the nondegeneracy of the dispersion matrix Σ was introduced in the literature by Harrison and Kreps [14] in order to ensure that the observable payoffs' process Y generates the (generally) unobservable Brownian filtration, the true underlying informational structure.[9] In this sense, the nonsingularity of Σ has been since regarded as fundamental and, given that it is equivalent to dynamic completeness when time is discrete, often conjectured to be related to dynamic completeness also in continuous time. A conjecture that, as Theorem 1 establishes formally, is correct in the strongest sense.

As a property, condition (ii) of the theorem depends only on the structure of the terminal dividends of the securities, leaving no role for the other economic primitives - in particular, the utility function of the representative

[7] The corresponding self-financing strategy is to buy-and-hold one unit of the bond. In bond units, this account pays no interest since its value is always one.

[8] For the case $K = 1$, dynamic completeness follows immediately from Theorem 1 in Diasakos [8]: as long as the now scalar σ is non-zero, the equilibrium relative price is always monotone in the current realization of the Brownian motion.

[9] I am referring to Proposition 1 in Harrison and Kreps [14] which allows also for Σ to be stochastic so that, for the K-dimensional Ito process Y, we would have here $\mathrm{d}\ln Y = \mu(Y(t), t)\,\mathrm{d}t + \Sigma(Y(t), t)\,\mathrm{d}\beta(\omega, t)$. As long as $\Sigma(\mathbf{y}, t)$ is nonsingular at every $(\mathbf{y}, t) \in \mathbb{R}^K \times [0, T]$, the Brownian filtration $\{\mathcal{F}_t : 0 \leq t \leq T\}$ is generated by $\{Y_t : 0 \leq t \leq T\}$.

agent or her endowment. It is also easily verified (by checking whether $|\Sigma| \neq 0$) and generically satisfied (within $\mathbb{R}^{2K}$, the set of points corresponding to singular square matrices is of zero-measure). And combining generic validity with universal verifiability is quite rare in the literature. In most generic results on dynamic completeness, the corresponding condition is shown to hold except for a small set of the primitive parameters, being nevertheless difficult (if not impossible in some cases) to establish whether it does for particular values of these parameters.

Indeed, Theorem 1 should be viewed in the context of the results on existence of general equilibrium in continuous-time finance models, established by a number of important papers (such as the ones mentioned already, but also Duffie and Skiadas [12]). To the best of my knowledge, apart from Raimondo [25] or Anderson and Raimondo [1]-[2], the typical approach has been to start with a given candidate equilibrium price process, which is assumed to be dynamically complete, and proceed to establish that it is in fact an equilibrium.[10] However, the candidate equilibrium price processes are determined from the economic primitives of the model (the utility functions of the agents, their endowments, and the dividend processes of the securities) by a fixed point argument. And this means that, except in the extremely special cases where one can solve for the candidate equilibrium explicitly, it is not possible to verify from the primitives if the candidate equilibrium price process is indeed dynamically-complete.

By contrast, Raimondo [25] as well as Anderson and Raimondo [2] allow for the case when the market is necessarily dynamically-incomplete ($N < K$). The latter being a direct extension of the former, both papers study a representative-agent economy and manage to construct the appropriate Negishi weights (hence, the equilibrium pricing process) directly from its primitives. Neither, however, specifies when the market is in fact dynamically complete, given that it is potentially so, while following their arguments is quite demanding as nonstandard analysis is heavily used.

Similar economic intuition and mathematical apparatus is deployed also

[10] Of course, the form of the assumption varies in the literature. See the introductory section of Anderson and Raimondo [1] for a summary review and discussion.

in Anderson and Raimondo [1]. Generalizing their previous work to an economy with many heterogeneous agents, the authors introduce here a condition on the economic primitives (in particular, on the dividends of the securities) which guarantees dynamic completeness, permitting the construction of the equilibrium pricing process via their representative agent approach. Yet, as I argue below, their economy embeds the one I examine in the present paper as a special case in which their condition reduces to the dispersion matrix Σ being nonsingular.

4 An Interpretation of the Result

Under the terminal dividend specification in (4), the partial derivative with respect to the current realization of the kth Brownian component at the node (ω, t) and for the realization $\mathbf{x} \in \mathbb{R}^K$ of the Brownian increment $\beta(\omega, T) - \beta(\omega, t)$ is given by $\frac{\partial D_n(\mathcal{I}(\omega,t),\mathbf{x})}{\partial \beta_k(\omega,t)} = \sigma_{nk} D_n(\mathcal{I}(\omega, t), \mathbf{x})$. As a consequence, the $K \times K$ Jacobian matrix of terminal dividends

$$J_D(\mathcal{I}(\omega, t), \mathbf{x}) = \begin{bmatrix} \nabla_{\beta(\omega,t)} D_1(\mathcal{I}(\omega, t), \mathbf{x})^{\intercal} \\ \cdot \\ \cdot \\ \nabla_{\beta(\omega,t)} D_K(\mathcal{I}(\omega, t), \mathbf{x})^{\intercal} \end{bmatrix}$$

is constructed by multiplying each row of Σ by the corresponding terminal dividend:

$$J_D(\mathcal{I}(\omega, t), \mathbf{x}) = [\sigma_n^{\intercal} D_n(\mathcal{I}(\omega, t), \mathbf{x})]_{n=1,\ldots,K} \tag{6}$$

It follows, therefore, that the nondegeneracy condition (ii) of the theorem is equivalent to requiring that $J_D(\beta(\omega, t), \mathbf{x})$ be of full rank at every node (ω, t) and for every realization $\mathbf{x} \in \mathbb{R}^K$ of the Brownian increment $\beta(\omega, T) - \beta(\omega, t)$. In other words, that $D_1(\mathcal{I}(\omega, t), \mathbf{x}), \ldots, D_K(\mathcal{I}(\omega, t), \mathbf{x})$ be locally linearly independent at every $(\omega, t, \mathbf{x}) \in \Omega \times [0, T) \times \mathbb{R}^K$.[11]

[11] Recall that, if the matrix $\widetilde{A}$ results from multiplying a row of the square matrix A by the number λ, then $|\widetilde{A}| = \lambda |A|$. In our case, $|J_D(\mathcal{I}(\omega, t), \mathbf{x})| = |\Sigma| \prod_{n=1}^{K} D_n(\mathcal{I}(\omega, t), \mathbf{x})$

Obviously, if there are just enough securities for potential dynamic completeness, some form of linear independence amongst the securities' dividends is necessary for dynamic completeness of the Arrow-Debreu securities prices. In this sense, some form of linear independence amongst the dividends is (at least implicitly) assumed in any paper within the realm of the continuous-time finance literature that deals with the case of potentially dynamically complete markets. Of course, not all of these papers have lump terminal dividends and not all present the corresponding form of dividend linear independence explicitly. In fact, to the best of my knowledge, only two recent papers do both explicitly. Namely, in Anderson and Raimondo [1] but also in Hugonnier et al. [16], dividend linear independence plays a pivotal role in guaranteeing dynamic completeness, the respective condition being equivalent to the one I present here for the corresponding setting.

Anderson and Raimondo [1] prove existence of equilibrium in a continuous-time securities market setting that embeds the one I examine. They also study a single consumption good, pure exchange economy in which the information- and time-structure for trade and consumption are exactly as here. Yet, their typical security may pay dividends even during the intermediate period, their economy has many heterogenous agents, while they allow for time- as well as state-dependence in the dividends, endowments, and instantaneous utilities (as long as the latter dependence obtains only through the realizations of the Brownian process.).

Their securities market is potentially dynamically complete for they introduce $K+1$ securities. In state ω, their typical security pays a dividend (measured in units of consumption) at some flow rate $d_n(\mathcal{I}(\omega,t))$ at times $t \in [0,T)$ and a lump amount $D_n(\mathcal{I}(\omega,T))$ at the terminal date. Their typical agent is endowed with the consumption good at some flow rate $e_i(\mathcal{I}(\omega,t))$ at times $t \in [0,T)$ and a lump amount $\rho_i(\mathcal{I}(\omega,T))$ at the end. Her preferences over consumption are given by a von Neumann-Morgenstern utility function U_i, such as the one I have considered in (3), in which the instantaneous utility functions v_i and u_i are defined on her measurable con-

with $\prod_{n=1}^{K} D_n(\mathcal{I}(\omega,t),\mathbf{x}) > 0$.

sumption process $c_i : [0,T] \times \Omega \mapsto \mathbb{R}_{++}$ but also on the process $\mathcal{I}$.

The authors take the functions that apply on flows, $d_n, e_i : \mathbb{R}^K \times [0,T) \mapsto \mathbb{R}_+$ and $v_i : \mathbb{R}_+ \times \mathbb{R}^K \times [0,T) \mapsto \mathbb{R} \cup \{-\infty\}$, to be analytic.[12] Regarding the ones that apply on lump amounts, $D_n, \rho_i : \mathbb{R}^K \times \{T\} \mapsto \mathbb{R}_+$ and $u_i : \mathbb{R}_+ \times \mathbb{R}^K \times \{T\} \mapsto \mathbb{R} \cup \{-\infty\}$, the first two are assumed to be continuous and the third twice continuously differentiable, all almost everywhere on their respective domains. In addition, the functions v_i and u_i are required to satisfy certain standard regularity conditions.

More importantly for the purposes of my analysis, Anderson and Raimondo [1] assume the following nondegeneracy condition on the terminal dividends: there exist (a) an open set $V \subset \mathbb{R}^K$ such that the terminal dividend of security 0 is positive if the terminal-date realization of the Brownian vector falls within this set ($D_0(\mathbf{y},T) > 0 \ \forall \mathbf{y} \in V$) and (b) some terminal-date Brownian realization $\mathbf{y}^* \in V$ such that the $K \times K$ Jacobian matrix $J_{D_0}(\mathbf{y},T)$ has full rank at $(\mathbf{y}^*,T)$.

$$J_{D_0}(\mathbf{y},T) = \begin{bmatrix} \nabla_{\mathbf{y}} \left(\frac{D_1(\mathbf{y},T)}{D_0(\mathbf{y},T)} \right)^{\intercal} \\ \cdot \\ \cdot \\ \nabla_{\mathbf{y}} \left(\frac{D_K(\mathbf{y},T)}{D_0(\mathbf{y},T)} \right)^{\intercal} \end{bmatrix} \qquad J_D(\mathbf{y},T) = \begin{bmatrix} \nabla_{\mathbf{y}} D_1(\mathbf{y},T)^{\intercal} \\ \cdot \\ \cdot \\ \nabla_{\mathbf{y}} D_K(\mathbf{y},T)^{\intercal} \end{bmatrix}$$

As it turns out, this exogenous assumption is sufficient for their equilibrium pricing process to be dynamically complete. As it happens, if the security 0 is a zero-coupon bond ($d_0(\mathcal{I}(\omega,t)) = 0$ for $t < T$ and $D_0(\mathcal{I}(\omega,T)) = 1$), this rank condition reduces to the requirement that the Jacobian matrix $J_D(\mathbf{y},T)$ is nonsingular at $(\mathbf{y}^*,T)$.

Needless to say, the economy I study in the present paper is a special case of the one just described. For it obtains by restricting the functions d_n and e_i to be, respectively, zero and deterministic (hence, either trivially analytic) while all terminal dividends in (4) are certainly continuous. Moreover, all conventional state-independent utility functions satisfy the conditions An-

[12] A function is said to be analytic if, at every point in its domain, there exists a power series which converges to the function on an open set containing the point.

derson and Raimondo impose on v_i and u_i. In fact, the authors themselves use the setting I analyze here as their main example (see their Section 3). It should be expected, therefore, that condition (ii) of Theorem 1 coincides with their nondegeneracy condition for the case in which security 0 is a zero-coupon bond.

Indeed, under the terminal dividend specification in (4), $D_n(\mathcal{I}(\omega, T)) = e^{\mu_n T + \sigma_n^{\intercal} \beta(\omega, T)}$ implies that the typical entry of the matrix $J_D(\mathbf{y}, T)$ reads $\sigma_{nk} D_n(\mathbf{y}, T)$, which is identical to the one in (6) once we define $\beta(\omega, T) = \mathbf{y} = \beta(\omega, t) + \mathbf{x}$. Hence, requiring the existence of some open set $V \subseteq \mathbb{R}^K$ and a point $\mathbf{y}^* \in V$ such that $J_D(\mathbf{y}^*, T)$ is nonsingular is equivalent to requiring that, conditional on the current realization $\beta(\omega, t)$, there exists some open set $V_{\beta(\omega,t)} \subseteq \mathbb{R}^K$ and a point $\mathbf{x}^* \in V_{\beta(\omega,t)}$ such that $J_D(\mathcal{I}(\omega, t), \mathbf{x}^*)$ is nonsingular. In the present formulation, however, when valid, this nondegeneracy condition remains so universally rather than at some point of an open set. In view of (6), the nondegeneracy of the Jacobian matrix of terminal dividends is globally equivalent to the nondegeneracy of Σ, a matrix of constants.

Of course, Anderson and Raimondo [1] study an economy with many heterogenous agents. By contrast, the analysis in the present paper is focused upon a representative agent. Yet, this does not matter with respect to a condition for dynamic completeness that is imposed on the structure of the securities' dividends only. It is well-known that, under dynamic completeness, the financial equilibrium must be Pareto optimal. Which, in an economy with many agents, requires in turn the existence of a (constant) vector of utility weights such that, at each node (ω, t), the equilibrium consumptions maximize the weighted sum of the utilities of the agents.

This weighted sum being the utility of the representative agent, Theorem 1 applies immediately because it is not concerned with the functional form of the weighted sum (or that of the social endowment). Hence, its claim remains in force even with many heterogenous agents. In this sense, it is not surprising that Theorem 1 gives the same sufficient condition for dynamic completeness (using, though, completely standard mathematical apparatus) as Anderson and Raimondo do when one of their securities is a zero-coupon

bond. What might be surprising perhaps (for reasons to become apparent immediately below) is that, in the important special case of their setup that I examine, it complements their nondegeneracy hypothesis by rendering it also necessary for dynamic completeness.

At first glance, this necessity might seem at odds with the study of Hugonnier et al. [16] who characterize dynamic completeness in a setting which is, in some dimensions more general than that in Anderson and Raimondo [1], in others more restrictive. On the one hand, the evolution of the state variable $\mathbf{y} \in \mathbb{R}^K$ need not be dictated by a Brownian motion but a general diffusion process. The time-horizon may also be infinite ($T = \infty$) so that a security pays a continuous flow of dividends but no terminal lump sum. Moreover, amongst the $K + 1$ traded securities, the zero security is taken to be a money market account. On the other hand, the agents' relative risk-aversion coefficients are taken to be bounded (see their Assumption C), in a way that is rather hard to satisfy unless their utility function exhibits constant relative risk aversion.

Following the usual methodology in the literature, the authors start with a given Arrow-Debreu equilibrium, construct then candidate equilibrium prices for the traded securities by using the aggregate consumption process as a state price deflator, and finally check that these are indeed equilibrium prices and, hence, support a dynamically-complete financial equilibrium. They establish that, as long as the candidate pricing process is real analytic as function of time, dynamic completeness can be deduced from the non-degeneracy of the volatility matrix of the dividends of the securities indexed by $1, \ldots, K$.[13]

[13] More precisely, the authors show that market completeness can be deduced from the primitives of the model as long as the candidate equilibrium prices of the securities are real analytic functions, of the state variables and time if the horizon is finite, and of time if it is infinite. In their technical supplement, however, they determine that, under some additional technical assumptions, real analyticity in the state variables can be dispensed with in the former case. By contrast, real analyticity in time cannot be relaxed. There are examples (see Section D of their supplement) of representative agent economies that fail to admit dynamically complete securities' markets, despite the fact that the dispersion matrix of their dividends is non-degenerate, because the candidate prices are not real analytic in time.

In particular, it is sufficient that there is some realization $\mathbf{y}^* \in \mathcal{Y}$ such that the Jacobian $J_D(\mathbf{y}^*, T)$ has full rank (see their Theorems 1 and 3). Of course, this condition is essentially the same as that in Anderson and Raimondo [1] for the corresponding setting (for an example, see Section B in the Supplement on Hugonnier et al. [16]). The authors argue, however, that is not also necessary for dynamic completeness. Indeed, if for instance some of their traded assets are fixed income securities (such as our zero-coupon bond or an annuity, its infinite-horizon analogue), non-singularity clearly fails (the respective row of the matrix $J_D(\cdot)$ is a row of zeroes) but the securities' market may nonetheless be dynamically complete (see their Example 1). In such cases, a sufficient criterion for dynamic completeness is obtained via a second order expansion of the volatility matrix of the prices (Theorems 2-3).

Yet, a fundamental assumption for the analysis in Hugonnier et al. [16] is that the instantaneous variance-covariance matrix of the underlying stochastic state-process is non-degenerate - see their Assumption A(a). And within the confines of the setting studied in the present paper (namely, that the state-variable follows a geometric Brownian motion), this amounts to requiring nothing other than condition (ii) of Theorem 1 above.[14] To the extent, therefore, that one adopts our normalization viewing essentially the zero-coupon bond as a money market account, dynamic completeness is effectively assumed on the outset.

Which is to say of course that, although very important in general, the extension undertaken by Hugonnier et al. [16] to introduce a money market account as an explicit security is rather meaningless in the particular economy the present paper analyzes. For if, in the presence of the zero-coupon bond, one of the securities indexed by $1, \ldots, K$ is riskless, the corresponding row of the matrix Σ will be a row of zeroes. It follows immediately then that condition (ii) would be violated, but so will be Assumption A(a). The same can be said, moreover, even when the model has an infinite time-horizon.

[14] To compare the notation used in the present paper with the one in Hugonnier et al. [16], their state-variable is given here by $X_t = \ln Y_t$. Hence, we have $\mu_X = \mu$ and $\sigma_X = \Sigma$ while their terminal dividend function $G_n(\cdot)$ is the exponential.

The preceding discussion applies also in that case because, as I show in the next section, Theorem 1 above continues to hold.

5 Proof of Theorem 1

The following result is borrowed from Diasakos [8] (see Lemma A.5 in Appendix A).

Lemma 5.1 *Let $S \subseteq \mathbb{R}^n$ be of non-zero Lebesgue measure and such that S^2 is symmetric around the origin.*[15] *Suppose also that*

(i) $g : S^2 \mapsto \mathbb{R}_+$ *is symmetric - i.e.,* $g(\mathbf{x}, \mathbf{y}) = g(\mathbf{y}, \mathbf{x})$ *- everywhere on its domain except for sets of measure zero,*[16]

(ii) $f : S^2 \mapsto \mathbb{R}$ *is such that* $f(\mathbf{x}, \mathbf{y}) + f(\mathbf{y}, \mathbf{x}) \geq 0$ *everywhere on its domain except for sets of measure zero, and*

(iii) $(gf)(\cdot)$ *is Lebesgue-integrable over* S^2.

Then

$$\int_{S^2} g(\mathbf{x}, \mathbf{y}) f(\mathbf{x}, \mathbf{y}) \, d(\mathbf{x}, \mathbf{y}) \geq 0$$

with strict inequality iff $g(\mathbf{x}, \mathbf{y}) [f(\mathbf{x}, \mathbf{y}) + f(\mathbf{y}, \mathbf{x})] \neq 0$ *on a subset of* S^2 *of non-zero measure.*

To keep notation simple, I will display neither the node (ω, t) of the Brownian filtration nor the process $\mathcal{I}$ as arguments in the relevant functions. Notice also that, even though not shown again for notational parsimony, all expectations are supposed to be conditional on the current filtration $\mathcal{F}_t$.

Let $N = K$. As argued in the main text, in the economy I examine, dynamic completeness is equivalent to the $K \times K$ matrix $J_p(\omega, t)$ being nonsingular almost everywhere on $\Omega \times [0, T]$. In what follows, I establish that the latter condition obtains iff Σ itself is nonsingular. To this end, $\mathbf{j}_{p,n}$

[15] This is to say that the relation $R(\mathbf{x}, \mathbf{y}) := \langle (\mathbf{x}, \mathbf{y}) \in \mathrm{S}^2 \rangle \subseteq \mathbb{R}^{2n}$ is symmetric.

[16] The lemma holds, more generally, if g is symmetric almost everywhere.

will denote the typical row of J_p in vector form. As shown in Diasakos [8] (see equation (27) in Appendix C), its typical entry $j_{p,n}^k = \frac{\partial p_n}{\partial \beta_k}$ is given by

$$\begin{aligned} j_{p,n}^k &= \frac{e^{\mu_n T + \sigma_n^\intercal \beta}}{P_0^2 \sqrt{T-t}\,(2\pi)^K} \\ &\quad \int_{\mathbb{R}^{2K}} u'\left(W\left(\mathbf{y}\right)\right) u'\left(W\left(\mathbf{x}\right)\right)\left(y_k - x_k\right) e^{\sqrt{T-t}\sigma_n^\intercal \mathbf{y} - \frac{\mathbf{y}^\intercal \mathbf{y} + \mathbf{x}^\intercal \mathbf{x}}{2}} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \end{aligned}$$

Only If. To establish the contrapositive statement, suppose that Σ is singular. There exists, then, $\mathbf{v} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$ s.t. $\Sigma \mathbf{v} = \mathbf{0}$. Take now $\mathbf{a} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$ s.t. $\mathbf{a}^\intercal \mathbf{v} \neq 0$ and consider the hyperplane $H_{\mathbf{a}} = \left\{\mathbf{x} \in \mathbb{R}^K : \mathbf{a}^\intercal \mathbf{x} = 0\right\}$. For an arbitrary $\mathbf{x}_0 \in H_{\mathbf{a}}$, consider also the line through $\mathbf{x}_0$ in the direction of $\mathbf{v}$: $L\left(\mathbf{x}_0; \mathbf{v}\right) = \left\{\mathbf{x} \in \mathbb{R}^K : \ \mathbf{x} = \mathbf{x}_0 + t\mathbf{v}, t \in \mathbb{R}\right\}$. Since $\mathbf{v}$ and $H_{\mathbf{a}}$ are not parallel, $\mathbb{R}^K$ can be spanned as $\cup_{\mathbf{x}_0 \in H_{\mathbf{a}}} L\left(\mathbf{x}_0; \mathbf{v}\right)$.[17] Hence, for the nth risky security, we have

$$\begin{aligned} \frac{P_0^2 \sqrt{T-t}\,(2\pi)^K}{e^{\mu_n T + \sigma_n^\intercal \beta}} \mathbf{a}^\intercal \mathbf{j}_{p,n} &= \int_{\mathbb{R}^{2K}} u'\left(F\left(\mathbf{x}\right)\right) u'\left(F\left(\mathbf{y}\right)\right) e^{\sigma_n^\intercal \mathbf{y}} \mathbf{a}^\intercal \left(\mathbf{y} - \mathbf{x}\right) e^{-\frac{\mathbf{y}^\intercal \mathbf{y} + \mathbf{x}^\intercal \mathbf{x}}{2}} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\ &= \int_{H_{\mathbf{a}}^2} S\left(\mathbf{x}_0, \mathbf{y}_0; \mathbf{a}\right) \mathrm{d}\mathbf{x}_0 \mathrm{d}\mathbf{y}_0 \end{aligned}$$

where $S : H_{\mathbf{a}}^2 \mapsto \mathbb{R}$ is given by

$$S\left(\mathbf{x}_0, \mathbf{y}_0; \mathbf{a}\right) = \int_{L^2(\mathbf{x}_0; \mathbf{v})} u'\left(F\left(\mathbf{x}\right)\right) u'\left(F\left(\mathbf{y}\right)\right) e^{\sigma_n^\intercal \mathbf{y}} \mathbf{a}^\intercal \left(\mathbf{y} - \mathbf{x}\right) e^{-\frac{\mathbf{y}^\intercal \mathbf{y} + \mathbf{x}^\intercal \mathbf{x}}{2}} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$$

But $\sigma_n^\intercal \mathbf{x} = \sigma_n^\intercal \mathbf{x}_0 \ \forall \mathbf{x} \in \text{Ł}\left(\mathbf{x}_0; \mathbf{v}\right)$ and $\forall n = 1, \ldots, N$ so that the terminal-period wealth is a function of $\mathbf{x}_0$ rather than $\mathbf{x}$ on $L\left(\mathbf{x}_0; \mathbf{v}\right)$. Moreover,

[17] Let $\{\mathbf{v}_k\}_{k=1}^{K-1}$ be a basis for the hyperplane $H_{\mathbf{a}}$. As it is not collinear with $\mathbf{v}$, $\{\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_{K-1}\}$ is a basis of $\mathbb{R}^K$. Hence, any $\mathbf{x} \in \mathbb{R}^K$ can be written uniquely as $\mathbf{x} = \sum_{k=1}^{K-1} t_k \mathbf{v}_k + t\mathbf{v}$ for some $(t, t_1, \ldots, t_{K-1}) \in \mathbb{R}^K$. Equivalently, $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ for a unique $\mathbf{x}_0 = \sum_{k=1}^{K-1} t_k \mathbf{v}_k \in H_{\mathbf{a}}$.

$\mathbf{a}^{\intercal}\mathbf{x} = t\mathbf{a}^{\intercal}\mathbf{v}\ \forall \mathbf{x} \in Ł\left(\mathbf{x}_0; \mathbf{v}\right)$. Hence, we may write

$$\begin{aligned}
& S\left(\mathbf{x}_0, \mathbf{y}_0; \mathbf{a}\right) \\
= & \int_{\mathbb{R}^2} u'\left(F\left(\mathbf{x}_0\right)\right) u'\left(F\left(\mathbf{y}_0\right)\right) e^{\sigma_n^{\intercal}\mathbf{y}_0}\left(t-\tau\right)\mathbf{a}^{\intercal}\mathbf{v} e^{-\frac{(\mathbf{y}_0+t\mathbf{v})^{\intercal}(\mathbf{y}_0+t\mathbf{v})+(\mathbf{x}_0+\tau\mathbf{v})^{\intercal}(\mathbf{x}_0+\tau\mathbf{v})}{2}} \mathrm{d}t\mathrm{d}\tau \\
= & u'\left(F\left(\mathbf{x}_0\right)\right) u'\left(F\left(\mathbf{y}_0\right)\right) e^{\sigma_n^{\intercal}\mathbf{y}_0} \int_{\mathbb{R}^2} \mathbf{a}^{\intercal}\left(t-\tau\right)\mathbf{v} e^{-\frac{(\mathbf{y}_0+t\mathbf{v})^{\intercal}(\mathbf{y}_0+t\mathbf{v})+(\mathbf{x}_0+\tau\mathbf{v})^{\intercal}(\mathbf{x}_0+\tau\mathbf{v})}{2}} \mathrm{d}t\mathrm{d}\tau \\
= & u'\left(F\left(\mathbf{x}_0\right)\right) u'\left(F\left(\mathbf{y}_0\right)\right) e^{\sigma_n^{\intercal}\mathbf{y}_0} \mathbb{E}_{(\mathbf{z},\widetilde{\mathbf{z}})}\left[\mathbf{a}^{\intercal}\left(\widetilde{\mathbf{z}}-\mathbf{z}\right)\right] \\
= & u'\left(F\left(\mathbf{x}_0\right)\right) u'\left(F\left(\mathbf{y}_0\right)\right) e^{\sigma_n^{\intercal}\mathbf{y}_0} \mathbf{a}^{\intercal}\left(\mathbf{x}_0-\mathbf{y}_0\right) = 0
\end{aligned}$$

where $(\mathbf{z}, \widetilde{\mathbf{z}}) \sim \mathcal{N}\left(-\left(\mathbf{x}_0, \mathbf{y}_0\right), \begin{pmatrix} \mathbf{v}\mathbf{v}^{\intercal} & \mathbf{0} \\ \mathbf{0} & \mathbf{v}\mathbf{v}^{\intercal} \end{pmatrix}\right)$. And as the choices of stock and Brownian node were arbitrary in this argument, we have just established that $\mathbf{a}^{\intercal}\mathbf{j}_{p,n}\left(\omega, t\right) = 0$ for all $n = 1, \ldots, K$ and all $(\omega, t) \in \Omega \times [0, T]$. The Jacobian $J_p\left(\omega, t\right)$ is indeed singular everywhere on $\Omega \times [0, T]$.

If. For any $\mathbf{v} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$, the non-singularity of Σ guarantees at least one nonzero entry for the vector $\Sigma\mathbf{v}$. Let it be the nth one: $\sigma_n^{\intercal}\mathbf{v} = \nu \neq 0$. Switching the vectors $\mathbf{a}$ and σ_n in the geometric argument made for the preceding part, we have

$$\frac{P_0^2\sqrt{T-t}\left(2\pi\right)^K}{e^{\mu_n T+\sigma_n^{\intercal}\beta}}\mathbf{v}^{\intercal}\mathbf{j}_n = \int_{H_{\sigma_n}^2} S\left(\mathbf{x}_0, \mathbf{y}_0; \mathbf{v}\right)\mathrm{d}\mathbf{x}_0\mathrm{d}\mathbf{y}_0$$

Yet, now $\mathbf{v}^\intercal\mathbf{x} = \mathbf{v}^\intercal\mathbf{x}_0 + t\mathbf{v}^\intercal\mathbf{v}$ and $\sigma_n^\intercal\mathbf{x} = t\nu\ \forall\mathbf{x} \in L(\mathbf{x}_0;\mathbf{v})$ and $\forall n = 1,\ldots,K$ so that

$$\begin{aligned}
& S(\mathbf{x}_0,\mathbf{y}_0;\mathbf{v}) \\
= \ & \mathbf{v}^\intercal(\mathbf{y}_0-\mathbf{x}_0) \\
& \int_{\mathbb{R}^2} u'(F(\mathbf{x}_0+\tau\mathbf{v}))\,u'(F(\mathbf{y}_0+t\mathbf{v}))\,e^{t\nu}e^{-\frac{(\mathbf{y}_0+t\mathbf{v})^\intercal(\mathbf{y}_0+t\mathbf{v})+(\mathbf{x}_0+\tau\mathbf{v})^\intercal(\mathbf{x}_0+\tau\mathbf{v})}{2}}\mathrm{d}t\mathrm{d}\tau \\
& +\mathbf{v}^\intercal\mathbf{v} \\
& \int_{\mathbb{R}^2} u'(F(\mathbf{x}_0+\tau\mathbf{v}))\,u'(F(\mathbf{y}_0+t\mathbf{v}))\,e^{t\nu}(t-\tau)\,e^{-\frac{(\mathbf{y}_0+t\mathbf{v})^\intercal(\mathbf{y}_0+t\mathbf{v})+(\mathbf{x}_0+\tau\mathbf{v})^\intercal(\mathbf{x}_0+\tau\mathbf{v})}{2}}\mathrm{d}t\mathrm{d}\tau \\
= \ & \mathbf{v}^\intercal(\mathbf{y}_0-\mathbf{x}_0)\,\mathbb{E}_{(\mathbf{z},\widetilde{\mathbf{z}})}\left[e^{\frac{\nu(\widetilde{z}_n-y_{0n})}{v_n}}u'(F(\mathbf{z}))\,u'(F(\widetilde{\mathbf{z}}))\right] \\
& +\mathbf{v}^\intercal\mathbf{v} \\
& \int_{\mathbb{R}^2} u'(F(\mathbf{x}_0+\tau\mathbf{v}))\,u'(F(\mathbf{y}_0+t\mathbf{v}))\,e^{t\nu}(t-\tau)\,e^{-\frac{(\mathbf{y}_0+t\mathbf{v})^\intercal(\mathbf{y}_0+t\mathbf{v})+(\mathbf{x}_0+\tau\mathbf{v})^\intercal(\mathbf{x}_0+\tau\mathbf{v})}{2}}\mathrm{d}t\mathrm{d}\tau
\end{aligned}$$

where $(\mathbf{z},\widetilde{\mathbf{z}}) \sim \mathcal{N}\left(-(\mathbf{x}_0,\mathbf{y}_0),\begin{pmatrix}\mathbf{v}\mathbf{v}^\intercal & \mathbf{0}\\ \mathbf{0} & \mathbf{v}\mathbf{v}^\intercal\end{pmatrix}\right)$.

There are two cases to consider, depending on whether or not $\mathbf{v}$ and σ_n are collinear. If they are, then $\mathbf{v}^\intercal\mathbf{x}_0 = \mathbf{v}^\intercal\mathbf{y}_0 = 0$. Otherwise, we can span $\mathbb{R}^K$ as $\cup_{r\in\mathbb{R}}\cup_{\mathbf{x}_0\in H_{(\sigma_n;r)}} L(\mathbf{x}_0;\mathbf{v})$ where $H_{(\sigma_n;r)} = \{\mathbf{x}_0 \in H_{\sigma_n} : \mathbf{v}^\intercal\mathbf{x}_0 = r\}$. In this case, we may write $\mathbf{v}^\intercal\mathbf{j}_{p,n} = \int_{\mathbb{R}}\left(\int_{H^2_{(\sigma_n;r)}} S(\mathbf{x}_0,\mathbf{y}_0;\mathbf{v})\,\mathrm{d}\mathbf{x}_0\mathrm{d}\mathbf{y}_0\right)\mathrm{d}r$. Regarding, though, the integration in the brackets, in the expansion for $S(\mathbf{x}_0,\mathbf{y}_0;\mathbf{v})$ above we now have $\mathbf{v}^\intercal(\mathbf{y}_0-\mathbf{x}_0) = r - r$.

In either case, therefore, $\mathbf{v}^\intercal(\mathbf{y}_0-\mathbf{x}_0) = 0$ and, thus,

$$\begin{aligned}
& S(\mathbf{x}_0,\mathbf{y}_0;\mathbf{v}) \\
= \ & \mathbf{v}^\intercal\mathbf{v} \\
& \int_{\mathbb{R}^2} u'(F(\mathbf{x}_0+\tau\mathbf{v}))\,u'(F(\mathbf{y}_0+t\mathbf{v}))\,e^{t\nu}(t-\tau)\,e^{-\frac{(\mathbf{y}_0+t\mathbf{v})^\intercal(\mathbf{y}_0+t\mathbf{v})+(\mathbf{x}_0+\tau\mathbf{v})^\intercal(\mathbf{x}_0+\tau\mathbf{v})}{2}}\mathrm{d}t\mathrm{d}\tau \\
= \ & \mathbf{v}^\intercal\mathbf{v} \\
& \int_{\mathbb{R}^2} u'(F(\mathbf{x}_0+t\mathbf{v}))\,u'(F(\mathbf{y}_0+\tau\mathbf{v}))\,e^{\tau\nu}(\tau-t)\,e^{-\frac{(\mathbf{y}_0+\tau\mathbf{v})^\intercal(\mathbf{y}_0+\tau\mathbf{v})+(\mathbf{x}_0+t\mathbf{v})^\intercal(\mathbf{x}_0+t\mathbf{v})}{2}}\mathrm{d}t\mathrm{d}\tau
\end{aligned}$$

Writing now $S(\mathbf{x}_0, \mathbf{y}_0; \mathbf{v})$ and $S(\mathbf{y}_0, \mathbf{x}_0; \mathbf{v})$ by the first and second of these equalities, respectively, gives

$$S(\mathbf{x}_0, \mathbf{y}_0; \mathbf{v}) + S(\mathbf{y}_0, \mathbf{x}_0; \mathbf{v}) = \mathbf{v}^\intercal \mathbf{v} \int_{\mathbb{R}^2} g(\mathbf{x}_0 + \tau\mathbf{v}, \mathbf{y}_0 + t\mathbf{v}) \left(e^{t\nu} - e^{\tau\nu}\right)(t - \tau)\, \mathrm{d}t\mathrm{d}\tau$$

with $g : \mathbb{R}^{2K} \mapsto \mathbb{R}_{++}$ defined by

$$g(\mathbf{x}_0 + \tau\mathbf{v}, \mathbf{y}_0 + t\mathbf{v}) = u'(F(\mathbf{x}_0 + \tau\mathbf{v}))\, u'(F(\mathbf{y}_0 + t\mathbf{v}))\, e^{-\frac{(\mathbf{y}_0 + t\mathbf{v})^\intercal (\mathbf{y}_0 + t\mathbf{v}) + (\mathbf{x}_0 + \tau\mathbf{v})^\intercal (\mathbf{x}_0 + \tau\mathbf{v})}{2}}$$

Now, since $\nu \neq 0$, $\nu\left(e^{t\nu} - e^{\tau\nu}\right)(t - \tau) > 0\ \forall t, \tau \in \mathbb{R}$ apart from the zero-measure subset (t, τ): $t = \tau$. Given also that $\mathbf{v}^\intercal \mathbf{v} > 0$, Lemma 5.1 implies that $S(\mathbf{x}_0, \mathbf{y}_0; \mathbf{v}) + I(\mathbf{y}_0, \mathbf{x}_0; \mathbf{v})$ has the same sign as ν. And so must do, of course, the quantity $2\mathbf{v}^\intercal \mathbf{j}_{p,n}$. Which proves that $J_p(\omega, t)$ is non-singular everywhere on $\Omega \times [0, T]$. For we have established that, at an arbitrary (ω, t) and for an arbitrary $\mathbf{v} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$, the vector $J_p(\omega, t)\,\mathbf{v}$ has at least one nonzero entry.

Dividend-financed Intermediate Consumption

Let us now turn to the setting where the agent's access to the trees' dividends is continuous so that her intermediate consumption can be financed by their payoffs at all times while T may also be infinite. In this case the equilibrium relative price is given by (2) and Diasakos [8] (see Appendix D) has established that, as long as $u(\cdot)$ and $D_n(\cdot)$ are, respectively, continuously-differentiable and continuous,

$$\begin{aligned}
P_n(t) &= \int_t^T P_{n,s}(t)\, \mathrm{d}s \\
\frac{\partial P_n(t)}{\partial \beta_k(t)} &= \int_t^T \frac{\partial P_{n,s}(t)}{\partial \beta_k(t)} \mathrm{d}s \qquad (n, k) \in \{0, 1, \ldots, N\} \times \{1, \ldots, K\} \\
P_0(t)^2 \frac{\partial p_n(t)}{\partial \beta_k(t)} &= \int_t^T P_{0,s}(t)^2 \frac{\partial p_{n,s}(t)}{\partial \beta_k(t)} \mathrm{d}s \qquad (n, k) \in \{1, \ldots, N\} \times \{1, \ldots, K\}
\end{aligned}$$

where $P_{n,s}(t)$ is the absolute price I have analyzed above, taking s to be the terminal date.

Recall now that the preceding proof is based upon showing that the inner product $\sum_{k=1}^{K} a_k \frac{\partial p_{n,s}(t)}{\partial \beta_k(t)}$ is either zero or not, for some $\mathbf{a} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$ and taking s as the terminal date. As the respective result applies also for the quantity $P_{0,s}(t)^2 \sum_{k=1}^{K} a_k \frac{\partial p_{n,s}(t)}{\partial \beta_k(t)}$ while the vector $\mathbf{a}$ is constant on $[t, T]$, the same is true for the quantity $P_0(t)^2 \sum_{k=1}^{K} a_k \frac{\partial p_n(t)}{\partial \beta_k(t)}$. That is, the inner product $\sum_{n=1}^{K} a_n \frac{\partial p_n(t)}{\partial \beta_k(t)}$ is correspondingly either zero or not so that the argument remains valid when intermediate consumption is dividend-financed. Needless to say, all of the above remain valid as $T \to \infty$.

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